On the behavior at infinity of solutions to difference equations in Schrödinger form

Evans M. Harrell II
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0610 USA
email: harrell@math.gatech.edu

Manwah Lilian Wong
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0610 USA
email: wong@math.gatech.edu

September 23, 2011

Abstract

We offer several perspectives on the behavior at infinity of solutions of discrete Schrödinger equations. First we study pairs of discrete Schrödinger equations whose potential functions differ by a quantity that can be considered small in a suitable sense as the index $n \to \infty$. With simple assumptions on the growth rate of the solutions of the original system, we show that the perturbed system has a fundamental set of solutions with the same behavior at infinity, employing a variation-of-constants scheme to produce a convergent iteration for the solutions of the second equation in terms of those of the original one. We use the relations between the solution sets to derive exponential dichotomy of solutions and elucidate the structure of transfer matrices.

Later, we present a sharp discrete analogue of the Liouville-Green (WKB) transformation, making it possible to derive exponential behavior at infinity of a single difference equation, by explicitly constructing a comparison equation to which our perturbation results apply. In addition, we point out an exact relationship connecting the diagonal part of the Green matrix to the asymptotic behavior of solutions. With both of these tools it is possible to identify an Agmon metric, in terms of which, in some situations, any decreasing solution must decrease exponentially.

A discussion of the discrete Schrördinger problem and its connection with orthogonal polynomials on the real line is presented in an Appendix.

1 Introduction

In this article we address the asymptotic behavior of solutions to linear difference equations of Schrödinger type, as the index n tends to infinity. We prove exponential dichotomy theorems and refined approximative expressions for the growing and subdominant (i.e., decaying) solutions, which have controlled errors.

We begin by approaching the subject as a perturbation analysis, showing that if two Schrödinger difference equations have potential terms that are sufficiently close, then they are asymptotically equivalent in the sense of [12], that is, there are solution bases for the two problems with the same behavior at infinity. The expressions obtained by the perturbation analysis are not merely asymptotic, but convergent for large but finite indices n. We follow with a classification of the possible asymptotic behaviors and some more estimates, including some cases where the asymptotic behavior of solutions does not match that of the comparison equation but can nonetheless be characterized.

Of course, when faced with one particular equation, comparison theorems are of limited use in the absence of a good equation to which one can compare. We therefore present some methods for constructing such equations after the perturbation analysis. Finally, we present some examples and remarks about connections with orthogonal polynomials.

Let Δ denote the discrete second-difference operator on the positive integer lattice. We standardize the Laplacian such that $(\Delta f)_n := f_{n+1} + f_{n-1} - 2f_n$ for $f = (f_n) \in \ell^2(\mathbb{N})$, and consider pairs of equations of the form

$$(-\Delta + V)\psi = 0, \tag{1.1}$$

$$(-\Delta + V^0)\phi = 0, \tag{1.2}$$

where the potential-energy functions V, V^0 are diagonal operators with real values V_n and V_n^0 respectively. (Complex V_n and V_n^0 could be allowed with, for the most part, only straightforward complications, but we prefer to keep the exposition focused.)

Our first aim is to find conditions under which the solutions of (1.1) have the same asymptotic behavior as $n \to \infty$ as those of the comparison (1.2) when the potential energies V and V_0 are close in a suitable sense. One application of the analysis is to the asymptotic behavior of eigenfunctions, in which case instead of (1.1) one could write

$$(-\Delta + V - E)\psi = 0 \tag{1.3}$$

and $(-\Delta + V^0 - E)\phi = 0$ for some real eigenvalue E. Again, for simplicity we shall absorb E into the definition of V, with no material restriction, because we consider the full set of solutions to (1.1) without restricting to eigensolutions of a particular realization of $-\Delta + V$ as an operator. Those interested in decay properties of eigenfunctions should systematically replace V_n in this article by $V_n - E$.

We do assume, however, that among the solution set of the comparison equation (1.2) there is a distinguished solution that decreases at infinity, unique up to a multiplicative constant, and we follow the nomenclature of ordinary differential equations in referring to such solutions as *subdominant*. (The term *recessive* is also frequently used.) We recall at this stage that if V has a constant value $V_\infty \notin [-4, 0]$, then explicit solutions are easily found, and it emerges that $(-\Delta - V_\infty)\phi = 0$ has a subdominant solution, indeed, one that decreases exponentially (see Example 6.1). Conversely, if $V = V_\infty \in [-4, 0]$, then there are no subdominant solutions. The significance of the interval $[-4, 0]$ is that it is the spectrum of Δ.

Remark 1. Equation (1.1) is invariant under the transformation

$$\psi_n \to (-1)^n \psi_n \tag{1.4}$$

$$V_n \to -4 - V_n, \tag{1.5}$$

as can be easily verified. Because of this, any fact proved under the assumption, for example, that $V_n > 0$ has a counterpart for $V_n < -4$. We shall use this remark to avoid repetition in some of our proofs.

When (1.2) has a subdominant solution, it will be denoted ϕ^- (fixing an overall constant), and ordinarily we shall identify a second, independent solution as ϕ^+. We recall that the Wronskian of two solutions of a discrete Schrödinger equation,

$$W[\phi_-, \phi_+] := \phi_n^- \phi_{n+1}^+ - \phi_{n+1}^- \phi_n^+, \tag{1.6}$$

is independent of the coordinate n, analogously to a well-known fact for Sturm-Liouville equations. (See, e.g., [1].) In terms of difference operators $\nabla^{\pm}$,

$$\nabla^{+} f_n := f_{n+1} - f_n \text{ and } \nabla^{-} f_n := f_n - f_{n-1}. \tag{1.7}$$

the Wronskian can also be expressed as

$$W = \phi_n^{-}(\nabla^{\pm}\phi_n^{+}) - \phi_n^{+}(\nabla^{\pm}\phi_n^{-}). \tag{1.8}$$

For future reference we recall some simple relations for the difference operators:

1. $\nabla^{+}(\nabla^{-} f_n) = \nabla^{-}(\nabla^{+} f_n) = \Delta f_n$;
2. (Chain Rule) $\nabla^{+}(fg)_n = (\nabla^{+} f_n)g_n + f_n(\nabla^{+} g_n) + (\nabla^{+} f_n)(\nabla^{+} g_n)$;
3. (Chain Rule) $\nabla^{-}(fg)_n = (\nabla^{-} f_n)g_n + f_n(\nabla^{-} g_n) - (\nabla^{-} f_n)(\nabla^{-} g_n)$.

In comparing (1.1) and (1.2) the behavior of solutions at infinity will be examined from several points of view. First, we study the asymptotic behavior of solutions under the effect of small perturbations of the potential as a fixed-point problem. We consider the solutions of (1.2) as known, and use them as the basis for a (convergent) variation-of-constants calculation of the solutions of (1.1). Then we introduce a factorization of the equation satisfied by the coefficients in that scheme in order to get a detailed understanding of how they converge.

Thereafter we present a new and efficient discrete variant of the Liouville-Green (WKB) approximation [22], so that for a given potential V a comparison equation (1.2) can be found for which the asymptotics are explicitly known, and consequently the behavior at infinity of solutions of (1.1) is explicitly determined, with controlled errors. As an alternative, following [11, 7, 8], we explore a set of related exact relations based on the diagonal of the Green matrix and their consequences for the behavior of solutions at infinity.

Finally, the reader may refer to the Appendix for a discussion of the relation between orthogonal polynomials and second-order difference equations. There the connection between ratio asymptotics of orthogonal polynomials and the results of Geronimo–Smith [13] will be discussed, and it will be shown how solutions of the discrete Schrödinger equation can be represented by orthogonal polynomials of the first and second kind.

We are far from the first to consider these questions, and like other researchers we mimic the better-developed theory known for Sturm-Liouville problems. Let us close the Introduction by placing our work in the context of the earlier literature.

A systematic study of certain difference equations dates from Poincaré [24]. In his work and in that of Birkhoff [4] asymptotic analysis was considered for equations using what would nowadays be termed transfer matrices of special types. A rather satisfactory understanding of the effect of small perturbations on stability questions for equations using transfer matrices, with dichotomy assumptions on their eigenvalues, was developed in [23, 10, 3], some of which is recounted in the monograph by Agarwal [1], which is a good source for showing how many of the standard facts from Sturm-Liouville theory can be ported over to the discrete setting, in particular, the technique of variation of constants. Coffman [10] and Benzaid–Lutz [3] studied product solutions, and in that regard prefigure in a rough way what we do in Section 4. The main results of [3] were discrete analogues of Levinson's fundamental lemma [20] for the asymptotic expression of the solution of a perturbed linear differential equation. In [3] the authors considered difference equations of the form

$$y(k+1) = [\Lambda(k) + R(k)]y(k). \tag{1.9}$$

Here $\Lambda(k)$ is an $N \times N$ diagonal matrix with non-zero diagonal entries $(\lambda_j(k))_{j=1}^N$ that satisfy a certain dichotomy condition. They further considered

$$x(k+1) = [\Lambda_0 + V(k) + R(k)]x(k), \tag{1.10}$$

again where Λ_0 is diagonal and V and R satisfy certain bounds. Their results apply widely to perturbed difference equations, but not readily to (1.1) and (1.2): As we shall see in (2.3), the transfer matrices in the present article are of the form

$$I + M_n = I + \frac{V_n - V_n^0}{W} \begin{pmatrix} \phi_n^+ \phi_n^- & \phi_n^- \phi_n^- \\ -\phi_n^+ \phi_n^+ & -\phi_n^+ \phi_n^- \end{pmatrix}, \tag{1.11}$$

which are neither diagonal nor diagonable if $V_n - V_n^0 \neq 0$. In fact, 1 is the only eigenvalue of the matrix $I + M_n$ and it has geometric multiplicity one. Moreover, the term $\frac{V_n - V_n^0}{W}(\phi_n^+)^2$ in the lower left corner typically diverges as $n \to \infty$.

Trench [28, 29] succeeded in giving conditions for the asymptotic equivalence of the solution sets of (1.1) and (1.2) in the sense considered by Hartman and Wintner [17], and seems to have been the first to realize that a good criterion for equivalence relies on an analysis of the expression

$$J_k := \phi_k^+ \phi_k^- \left(V_k - V_k^0\right), \tag{1.12}$$

(in our notation). In [9], following Trench, a necessary and sufficient condition for asymptotic equivalence for some difference equations related to (1.1) and (1.2) is spelled out in terms of J. Although we bring different methods to bear on asymptotic equivalence in the following sections, ℓ^p norms of (1.12) and similar quantities remain central; see Theorems 2.2, 3.2, and 3.3. One could interpret these norms as traces of operator perturbations like those occurring in studies of spectral–shift functions (e.g., see [15]), leading us to speculate that direct connections between the spectral-shift functions and behavior at infinity could be found.

After a discussion of asymptotic equivalence, we take advantage of the specific Schrödinger form of the equation, and construct comparison equations having product solutions of a certain structure, inspired by the classical Liouville-Green, or WKB, approximation. Of prior work on discrete versions of the Liouville-Green approximation we single out that of Geronimo and Smith [13], which was inspired by some earlier work of Braun [5]. Geronimo and Smith studied a somewhat more general equation than (1.1),

$$d_{n+1} y_{n+1} - q_n y_n + y_{n-1} = 0, \tag{1.13}$$

where d_n and q_n are sequences of numbers with $d_n \neq 0$ for $n = 1, 2, \ldots$, and pursued a Riccati analysis for solutions in product form. In Section 4 we identify a more explicit and efficient product scheme along the lines of the Liouville-Green approximation as presented in [22], to which we apply the perturbation analysis developed in Section 2. Yet another article with Liouville-Green analysis using products is [6], in which an explicit semiclassical parameter appears, and the Green matrix is studied in a product form and used to prove refined stability results for nonhomogeneous difference equations. In the following subsection we relate the discrete Liouville-Green approximation to the diagonal of the Green function, following ideas pioneered in [11], which have previously been somewhat developed in the study of difference equations by Chernyavskaya and Shuster [7, 8, 6].

2 Variation of constants and behavior at infinity

We begin by casting the problem of understanding the asymptotic dependence of solutions at infinity as a problem on a certain weighted Banach space, following ideas of [16] in the continuous case,

which was in turn inspired by [17]. Suppose that V is close to another potential V^0 such that the solutions to $(-\Delta + V^0)\phi = 0$ are understood, in the sense that a pair of independent solutions $\phi^\pm$ can be identified, including a subdominant solution $\phi_n^- \in \ell^2$. A perturbation analysis can be based on the following way of connecting the solutions of (1.1) and (1.2).

Theorem 2.1. *Let V and V^0 be two potential functions, and let $\phi^\pm$ be independent solutions to the equation* (1.2). *We may represent any ψ as a linear combination of $\phi^\pm$ with variable coefficients $a_n^\pm$, i.e.,*

$$\psi_n = a_n^+ \phi_n^+ + a_n^- \phi_n^-. \tag{2.1}$$

Then ψ is a solution to the equation (1.1) *if and only if we may find sequences $(a_n^\pm)_{n=1}^\infty$ that satisfy the following two conditions: For all $n \geq 1$,*

$$(\nabla^- a_n^+)\phi_{n-1}^+ + (\nabla^- a_n^-)\phi_{n-1}^- = 0, \tag{2.2}$$

$$\begin{pmatrix} a_{n+1}^+ \\ a_{n+1}^- \end{pmatrix} = \left[I + \frac{V_n - V_n^0}{W} \begin{pmatrix} \phi_n^+\phi_n^- & \phi_n^-\phi_n^- \\ -\phi_n^+\phi_n^+ & -\phi_n^-\phi_n^+ \end{pmatrix} \right] \begin{pmatrix} a_n^+ \\ a_n^- \end{pmatrix} =: (I + M_n) \begin{pmatrix} a_n^+ \\ a_n^- \end{pmatrix}, \tag{2.3}$$

under the convention that $\phi_0^\pm = 0$.

Proof of Theorem 2.1. Suppose that ψ is a solution to (1.1). Since the expression (2.1) has two degrees of freedom we have the liberty to impose a second condition on the coefficients to so that

$$\nabla^- \psi_n = a_n^+(\nabla^- \phi_n^+) + a_n^-(\nabla^- \phi_n^-). \tag{2.4}$$

Observe that (2.4) implicitly sets the following expression to zero

$$(\nabla^- a_n^+)\phi_n^+ + (\nabla^- a_n^-)\phi_n^- - (\nabla^- a_n^+)(\nabla^- \phi_n^+) - (\nabla^- a_n^-)(\nabla^- \phi_n^-) = 0; \tag{2.5}$$

by direct expansion (2.5) is equivalent to (2.2).

Now compute $\Delta\psi_n = \nabla^+\nabla^-\psi_n$ based on the expression (2.4). By the chain rules for $\nabla^\pm$,

$$\begin{aligned} \Delta\psi_n = (\nabla^+ a_n^+)(\nabla^-\phi_n^+) + a_n^+(\Delta\phi_n^+) + (\nabla^+ a_n^+)(\Delta\phi_n^+) \\ + (\nabla^+ a_n^-)(\nabla^-\phi_n^-) + a_n^-(\Delta\phi_n^-) + (\nabla^+ a_n^-)(\Delta\phi_n^-). \end{aligned} \tag{2.6}$$

By substituting $\Delta\psi_n = V_n\psi_n$ and $\Delta\phi_n^\pm = V_n^0\phi_n^\pm$ into (2.6), we obtain

$$(V_n - V_n^0)(a_n^+\phi_n^+ + a_n^-\phi_n^-) = (\nabla^-\phi_n^+ + V_n^0\phi_n^+)(\nabla^+ a_n^+) + (\nabla^-\phi_n^- + V_n^0\phi_n^-)(\nabla^+ a_n^-). \tag{2.7}$$

In order for the coefficients of $\nabla^+ a_n^-$ in (2.7) and (2.2) to match, we multiply (2.7) by ϕ_n^- and (2.2) by $(\nabla^-\phi_n^- + V_n^0\phi_n^-)$. Then we subtract one from the other and get

$$(\nabla^+ a_n^+)W = (V_n - V_n^0)\left(\phi_n^+ a_n^+ + \phi_n^- a_n^-\right)\phi_n^- = (V_n - V_n^0)\psi_n\phi_n^-, \tag{2.8}$$

where W is the Wronskian as defined in (1.6).

Similarly, we match the coefficients of $\nabla^+ a_n^-$ in (2.7) and (2.2) by multiplying (2.7) with ϕ_n^+ and (2.2) with $(\nabla^-\phi_n^+ + V_n^0\phi_n^+)$. Then we obtain

$$(\nabla^+ a_n^-)(-W) = (V_n - V_n^0)\left(\phi_n^+\phi_n^+ a_n^+ + \phi_n^-\phi_n^+ a_n^-\right) = (V_n - V_n^0)\psi_n\phi_n^+. \tag{2.9}$$

Putting (2.8) and (2.9) together in matrix form, we arrive at (2.3).

For the implication in the other direction, suppose that the sequences $a_n^\pm$ satisfy (2.3) and (2.2). By direct expansion of (2.1), we find that

$$\Delta\psi_n = V_n^0\psi_n + (\text{right side of (2.7)}) + \nabla^+ (\text{left side of (2.2)}) . \tag{2.10}$$

With (2.3), (2.8) and (2.9) follow. If we now apply these relations to the right side of (2.7), it becomes

$$\frac{V_n - V_n^0}{W} \left[(\nabla^-\phi_n^+ + V_n^0\phi_n^+)\psi_n\phi_n^- - (\nabla^-\phi_n^- + V_n^0\phi_n^-)\psi_n\phi_n^+\right], \tag{2.11}$$

which simplifies to

$$\psi_n\frac{V_n - V_n^0}{W}[\phi_n^-\nabla^-\phi_n^+ - \phi_n^+\nabla^-\phi_n^-] = (V_n - V_n^0)\psi_n. \tag{2.12}$$

Returning to (2.10), we conclude that $\Delta\psi_n = V_n\psi_n$ for all n only if $\nabla^+\left((\nabla^- a_n^+)\phi_{n-1}^+ + (\nabla^- a_n^-)\phi_{n-1}^-\right) = 0$ for all n, or in other words, when the expression on the left side of (2.2) is a constant. Finally, note that since its value is zero when $n = 1$, it is zero for all n. $\square$

2.1 Convergence of $a_n^\pm$ in a suitable Banach space

An advantage of the variation-of-constants approach to asymptotic equivalence over the methods of [28, 28, 9] is that it provides a rapidly convergent iterative scheme with error estimates that can be made explicit. To set it up, we introduce the notation

$$\mathbf{a}_n = \begin{pmatrix} a_n^+ \\ a_n^- \end{pmatrix} \quad \text{and} \quad \beta_n = \frac{V_n - V_n^0}{W}, \tag{2.13}$$

where W is the Wronskian as in (1.6). We shall regard $\mathbf{a}$ as an element of a weighted Banach space,

$$\mathcal{B}_N := \left\{ \mathbf{X} = \begin{pmatrix} X_n^+ \\ X_n^- \end{pmatrix} : \|\mathbf{X}\|_N := \sup_{n\geq N} \left(|(\phi_n^+)^2 X_n^+| + |X_n^-|\right) < \infty \right\}. \tag{2.14}$$

Substituting the expression for ψ into (2.3), we calculate

$$(\nabla^-\mathbf{a}_n) = \beta_n \begin{pmatrix} \phi_n^+\phi_n^- & (\phi_n^-)^2 \\ -(\phi_n^+)^2 & -\phi_n^+\phi_n^- \end{pmatrix} \mathbf{a}_n, \tag{2.15}$$

or, by summing (2.15),

$$\begin{aligned} \mathbf{a}_n &= \mathbf{a}_{n+1} - \beta_n \begin{pmatrix} \phi_n^+\phi_n^- & (\phi_n^-)^2 \\ -(\phi_n^+)^2 & -\phi_n^+\phi_n^- \end{pmatrix} \mathbf{a}_n \\ &= \dots \\ &= \mathbf{a}_{n+\ell} - \sum_{k=n}^{n+\ell-1} \beta_k \begin{pmatrix} \phi_k^+\phi_k^- & (\phi_k^-)^2 \\ -(\phi_k^+)^2 & -\phi_k^+\phi_k^- \end{pmatrix} \mathbf{a}_k. \end{aligned}$$

Formally letting $\ell \to \infty$, $a_{n+\ell} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we therefore define a linear operator $\mathcal{M}$ by

$$(\mathcal{M}\mathbf{X})_n := \sum_{k=0}^{\infty} \beta_{n+k} \begin{pmatrix} \phi_{n+k}^+\phi_{n+k}^- & (\phi_{n+k}^-)^2 \\ -(\phi_{n+k}^+)^2 & -\phi_{n+k}^+\phi_{n+k}^- \end{pmatrix} \mathbf{X}_{n+k}. \tag{2.16}$$

The convergence of the coefficients in the Banach space proceeds as follows:

Theorem 2.2. *Suppose that (1.2) has a solution basis $\phi^{\pm}$ such that $\lim_{n\to\infty}\phi_n^- = 0$, $|\phi_n^+|$ is monotonically nondecreasing for sufficiently large n, and β_n (cf. (2.13)) satisfies $\beta_n(1+|\phi_n^+\phi_n^-|^2) \in \ell^1$. Then for N sufficiently large, $\mathcal{M}$ is a contraction on $\mathcal{B}_N$. Consequently, there exists a unique solution ψ^- of (1.1) such that*

$$\psi_n^- = a_n^+\phi_n^+ + a_n^-\phi_n^-,$$

where $\lim_{n\to\infty} a_n^+ = 0$ and $\lim_{n\to\infty} a_n^- = 1$. Moreover, if we define $\widehat{\psi}_n^- := \max_{m\geq n}|\phi_n^-|$, then

$$\psi_n^- = \phi_n^- + r_n\widehat{\psi}_n^-, \tag{2.17}$$

with $\lim_{n\to\infty} r_n = 0$.

Remark 2. If $|\phi_n^-|$ is monotone nonincreasing, then we may simply write $\psi_n^- = (1+r_n)\phi_n^-$, with $\lim_{n\to\infty} r_n = 0$. If the product $\phi_n^+\phi_n^-$ is bounded, it suffices for this theorem to assume that $\beta_n \in \ell^1$; circumstances under which this is guaranteed are discussed below in Lemma 2.21, Theorem 4.3 and Theorem 4.2.

Proof. For N sufficiently large, we claim that $\mathcal{M}$ is a strict contraction on $\mathcal{B}_N$. To see this, we introduce the shorthand $\sup\left|\begin{pmatrix} X_n^+ \\ X_n^- \end{pmatrix}\right|$ for $\sup_{n\geq N}(|X_n^+| + |X_n^-|)$, and observe that $\|\mathbf{X}\|_{\mathcal{B}_N} = \sup_{m\geq N}\left|\begin{pmatrix} (\phi_m^+)^2 & 0 \\ 0 & 1\end{pmatrix}\mathbf{X}_m\right|$. Thus

$$\begin{aligned}
\|\mathcal{M}\mathbf{X}\|_{\mathcal{B}_N} &= \sup_{n\geq N}\left|\begin{pmatrix} |\phi_n^+|^2 & 0 \\ 0 & 1\end{pmatrix}\sum_{k=0}^{\infty}\beta_{n+k}\begin{pmatrix}\phi_{n+k}^+\phi_{n+k}^- & (\phi_{n+k}^-)^2 \\ -(\phi_{n+k}^+)^2 & -\phi_{n+k}^+\phi_{n+k}^-\end{pmatrix}\mathbf{X}_{n+k}\right| \\
&= \sup_{n\geq N}\left|\sum_{k=0}^{\infty}\beta_{n+k}\begin{pmatrix} |\phi_n^+|^2 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}\phi_{n+k}^+\phi_{n+k}^- & (\phi_{n+k}^-)^2 \\ -(\phi_{n+k}^+)^2 & -\phi_{n+k}^+\phi_{n+k}^-\end{pmatrix}\times\right. \\
&\qquad\qquad \left.\begin{pmatrix} |\phi_{n+k}^+|^2 & 0 \\ 0 & 1\end{pmatrix}^{-1}\begin{pmatrix} |\phi_{n+k}^+|^2 & 0 \\ 0 & 1\end{pmatrix}\mathbf{X}_{n+k}\right| \\
&= \sup_{n\geq N}\left|\sum_{k=0}^{\infty}\beta_{n+k}\begin{pmatrix}\phi_{n+k}^+\phi_{n+k}^-\left|\phi_n^+/\phi_{n+k}^+\right|^2 & (\phi_{n+k}^+\phi_{n+k}^-)^2\left|\phi_n^+/\phi_{n+k}^+\right|^2 \\ -1 & -\phi_{n+k}^+\phi_{n+k}^-\end{pmatrix}\begin{pmatrix} |\phi_{n+k}^+|^2 & 0 \\ 0 & 1\end{pmatrix}\mathbf{X}_{n+k}\right| \\
&\leq \|\mathbf{X}\|_{\mathcal{B}_N}\sup_{n\geq N}\sum_{k=0}^{\infty}|\beta_{n+k}|\max(|\phi_{n+k}^+\phi_{n+k}^-| + |\phi_{n+k}^+\phi_{n+k}^-|^2, 1+|\phi_{n+k}^+\phi_{n+k}^-|) \\
&\leq 2\|\mathbf{X}\|_{\mathcal{B}_N}\sum_{k=0}^{\infty}|\beta_{N+k}|\,(1+|\phi_{N+k}^+\phi_{N+k}^-|^2)
\end{aligned}$$

We then ask whether there is a solution to

$$\mathbf{a}_n = \begin{pmatrix} 0 \\ 1\end{pmatrix} - (\mathcal{M}\mathbf{a})_n,$$

and conclude by the contraction mapping theorem that there is, for N large enough that

$$\sum_{k=0}^{\infty}|\beta_{N+k}|\,(1+|\phi_{N+k}^+\phi_{N+k}^-|^2) < \frac{1}{2}.$$

Indeed, therefore $(\mathbf{1}+\mathcal{M})\mathbf{a} = \begin{pmatrix}0\\1\end{pmatrix}$ is uniquely solved by the norm-convergent Neumann series

$$\mathbf{a} = \left(\sum_{\ell=0}^{\infty}(-\mathcal{M})^{\ell}\right)\begin{pmatrix}0\\1\end{pmatrix}. \tag{2.18}$$

Being dominated by a geometric series, the convergence of (2.18) is exponentially fast.

For the final statement, we need a lemma about the operator $\mathcal{M}$:

Lemma 2.3. *Suppose that* $|x_n^+| \le C_1|\phi_n^-|^2$ *and* $|x_n^-| \le C_2$. *Then*

$$|\mathcal{M}x|_n^+ \le M\ (C_1+C_2)|\phi_n^-|^2 \tag{2.19}$$

and

$$|\mathcal{M}x|_n^- \le M\ (C_1+C_2), \tag{2.20}$$

where $M := \|\beta_n(1+|\phi_n^+\phi_n^-|^2)\|$.

The lemma is an easy estimate from the definition of $\mathcal{M}$.

The proof of the final statement of the theorem then requires choosing N sufficiently large that the coefficients in the conclusions of the lemma are small enough that $\mathcal{M}$ is a contraction, and summing the Neumann series (2.18). □

2.2 Construction of a second solution and estimates of the product of the two solutions

Since understanding the asymptotic behavior of the solutions of the perturbed equation requires knowledge of a full set of independent solutions $\{\phi_n^+, \phi_n^-\}$ to the original equation, we recall a standard reduction-of-order formula showing that the subdominant solution determines a second, independent solution, which grows at infinity. (E.g., the text [17] treats this argument in the continuous case in §XI.2, and it can be found in the discrete literature in numerous places, including [28, 9].) The following simple formula does not require a subdominant solution, only one that is nonvanishing. It is true by direct verification that ψ_n^+ solves (1.1) and that $W=1$ when $n=m$. (Of course it is derived by positing that $\psi_n^+ = \gamma_n\psi_n^-$, substituting, and using the Wronski identity to determine γ_n.)

Lemma 2.4. *(Standard) Suppose that* (1.1) *has a solution that is nonzero for all* n, $n = m,\dots,M$. *If*

$$\psi_n^+ := \begin{cases} 0 & \text{if } n=m \\ \psi_n^- \displaystyle\sum_{k=m}^{n-1}\frac{1}{\psi_k^-\psi_{k+1}^-} & \text{if } n>m. \end{cases} \tag{2.21}$$

then ψ_n^+ *is an independent solution of* (1.1) *on the interval* $[m,M]$.

The lemma has some simple but useful consequences:

Corollary 2.5. *Suppose that for some* a, $|a|>1$, (1.1) *has a solution such that* $a^n\psi_n^- \to 1$ *as* $n\to\infty$. *Then*

- *Every solution* ψ_n *of* (1.1) *that is independent of* ψ_n^- *is exponentially increasing, i.e.,* $a^{-n}\psi_n \to C \neq 0$ *as* $n\to\infty$.

- *For any solution ψ_n, the product $\psi_n \psi_n^-$ is bounded independently of n.*

- *Given any boundary condition of the form $a\psi_1 + b\psi_2 = 0$, if $0 \notin \mathrm{sp}(-\Delta + V)$, then the Green matrix for $(-\Delta + V)$ on $n \geq 1$ is uniformly bounded.*

Proof. Because every solution to (1.1) is a linear combination of ψ_n^- and ψ_n^+ as defined in (2.21), it suffices to show the first two statements for $\psi_n = \psi_n^+$, which behaves asymptotically like

$$a^{-n} \sum_{k}^{n-1} a^{2k+1}. \tag{2.22}$$

Since this geometric series can be bounded above and below by $(C_1 + C_2\, a^{-n} \int_1^{n-1} a^{2x+1} dx = \frac{C_2}{\ln a} a^{n-1} + O(1))\ C_1 + C_2 a^n$ the first two statements follow.

If $0 \neq \mathrm{sp}(-\Delta + V)$, then the Green matrix is defined, and

$$G_{mn} = \frac{\psi^+_{\min(m,n)} \psi^-_{\max(m,n)}}{W[\psi^-, \psi^+]}, \tag{2.23}$$

where ψ_n^+ satisfies the boundary condition at $n = 1, 2$. Since this is bounded on any finite set of indices m, n, the third statement follows from the asymptotic estimate of ψ_n^+ in the first statement. □

An important case where these estimates apply is captured in the following.

Corollary 2.6. *Suppose that for some constant $V_\infty \notin [-4, 0]$, $V - V_\infty \in \ell^1$. Then there is a solution to* (1.1) *of the type $\psi_n^- \sim a^{-n}$ for $|a| > 1$ and an independent solution $\psi_n^+ \sim a^n$, and the statements of Corollary* 2.5 *apply.*

Proof. We can apply Theorem 2.2 and Corollary 2.5 once it is observed that the comparison equation

$$(-\Delta + V_\infty)\phi_n = 0$$

has an exponentially decreasing solution, *viz.*, assuming $V_\infty > 0$, $\phi_n^- = a^{-n}$ for $a = \frac{1}{2}\left(V_\infty + \sqrt{V_\infty^2 + 4V_\infty}\right)$. (The case $V_\infty < -4$ similarly has an exponentially decreasing solution, according to Remark 1.) □

Further conditions for the existence of exponentially decreasing and exponentially increasing solutions may be found in [32].

The existence of a more rapidly decreasing solution has similar implications:

Corollary 2.7. *Suppose that for some $a > 1, b > 1$, equation* (1.1) *has a solution such that $a^{n^b} \psi_n^- \to 1$ as $n \to \infty$. Then*

- *Every solution ψ_n of* (1.1) *that is independent of ψ_n^- increases rapidly as $n \to \infty$, but $a^{-n^b} \psi_n \to 0$ as $n \to \infty$.*

- *For any solution ψ_n, the product $\psi_n \psi_n^-$ is bounded independently of n.*

- *Given any boundary condition of the form $a\psi_1 + b\psi_2 = 0$, if $0 \neq \mathrm{sp}(-\Delta + V)$, then the Green matrix for $(-\Delta + V)$ on $n \geq 1$ is uniformly bounded.*

The proof of this is similar to that of Corollary 2.5, but details will be left to the interested reader. On the other hand, subdominant solutions that decrease only polynomially fast do not lead to as strong control of the products or of the Green matrix:

Corollary 2.8. *Suppose that for some $a > 0$ Equation* (1.1) *has a solution such that $n^a \psi_n^- \to 1$ as $n \to \infty$. Then for any solution ψ that is independent of ψ_n^-, $|\psi_n^- \psi_n| \sim Cn$.*

Proof sketch. The argument being familiar from the proof of Corollary 2.5, we content ourselves with the application of Formula (2.21). As before, we may as well assume that $\psi_n = \psi_n^+$ as defined by that formula, which is of the form

$$n^{-a} \sum_{k}^{n-1} k^a (k+1)^a. \tag{2.24}$$

This is asymptotic to $n^{-a} \int_1^{n-1} x^{2a} dx = \frac{n^{a+1}}{2a+1} + o(n^{a+1})$. The claimed estimate for the product results when this is multiplied by ψ_n^-. □

Some converse implications, by which the boundedness of $\phi_n^+ \phi_n^-$ controls the asymptotic behavior of solutions, will appear in Section 4.

3 Refined asymptotic estimates

Let ψ be a non-trivial solution to the equation $-\Delta + V = 0$, when $V_n^0 - V_n$ is small, and $\psi_n = a_n^+ \phi_n^+ + a_n^- \phi_n^-$. In this section, we provide a classification of the parameters a_n^+ and a_n^- and describe how they converge.

To begin, we prove a preliminary classification of a^+ and a^-, distinguishing the exceptional cases where the perturbed solutions only depend on one of the comparison solutions in (2.1) for large n:

Proposition 3.1 (primary classification)**.** *Suppose $\beta_n \phi_n^+ \phi_n^- \to 0$. Then for any non-trivial solution $\psi_n = a_n^+ \phi_n^+ + a_n^- \phi_n^-$ with $a_n^\pm$ satisfying* (2.3)*, one of the following must be true:*

1. *Given any integer N, there is an integer $p > N$ such that both a_p^+ and a_p^- are non-zero (this will be treated in Theorem* 3.2 *below).*
2. *There is an integer p_0 such that*

$$a_{p_0+k}^+ = a_{p_0}^+ \neq 0, \quad a_{p_0+k}^- = 0 \quad \forall k \geq 0. \tag{3.1}$$

3. *There is an integer p_0 such that*

$$a_{p_0+k}^- = a_{p_0}^- \neq 0, \quad a_{p_0+k}^+ = 0 \quad \forall k \geq 0. \tag{3.2}$$

Proof of Proposition 3.1*.* Recall the definition in (2.3). Observe that $\det M_p = \mathrm{Tr} M_p = 0$. Hence, for all $p \in \mathbb{N}$,

$$\det(I + M_p) = 1 + \mathrm{Tr} M_p + \det M_p = 1, \tag{3.3}$$

which implies that

$$\mathbf{a}_n := \begin{pmatrix} a_n^+ \\ a_n^- \end{pmatrix} = \mathbf{0} \text{ for some } m \quad \Leftrightarrow \quad \mathbf{a}_p = \mathbf{0}, \forall p \in \mathbb{N} \quad \Leftrightarrow \quad \mathbf{a}_1 = \mathbf{0}. \tag{3.4}$$

Since ψ is not the trivial solution, for any p either $a_p^+ \neq 0$ or $a_p^- \neq 0$ (or both).

Suppose we are not in Case 1. Then there exists N_0 such that for all $k \geq N_0$, either $a_k^+ = 0$ or $a_k^- = 0$. Without loss of generality, suppose for some large $p > N_0$, $a_p^+ \neq 0$ and $1 + \beta_k \phi_k^+ \phi_k^- \neq 0$ for all $k \geq p$ (this is possible because $\beta_k \phi_k^+ \phi_k^- \to 0$). We will show that this corresponds to Case 2.

Observe that

$$\begin{pmatrix} a_{p+1}^+ \\ a_{p+1}^- \end{pmatrix} = (I + M_p) \begin{pmatrix} a_p^+ \\ 0 \end{pmatrix} = a_p^+ \begin{pmatrix} 1 + \beta_p \phi_p^+ \phi_p^- \\ -\beta_p (\phi_p^+)^2 \end{pmatrix}, \tag{3.5}$$

which implies $a_{p+1}^+ \neq 0$ and as a result, $a_{p+1}^- = 0$. Apply the same argument recursively to obtain (3.1) $\square$

Assumptions 3.1. Now we focus on Case 1 of Proposition 3.1. Without loss of generality, we assume $a_1^+, a_1^- \neq 0$ and $\sup_n |\beta_n \phi_n^+ \phi_n^-| < 1$. The latter assumption, together with the assumption in Theorem 3.2 that $\sum_{n=1}^\infty |\beta_n \phi_n^+ \phi_n^-| < \infty$, guarantees that $p_n^\pm \neq 0$ for all n and that $\lim_{n\to\infty} \Pi_n^\pm \neq 0$.

A key observation here is that the recurrence matrix $I + M_n$ in (2.3) can not always be diagonalized, making it impossible to utilize existing techniques in the perturbation theory literature, which heavily relies on the fact that the transfer matrix can be diagonalized (see, e.g., Benzaid–Lutz [3] and the discussion in the Introduction).

A key observation is that the recurrence matrix can be decomposed into the sum of a lower triangular matrix and an upper triangular error matrix:

$$I + M_n = \begin{pmatrix} 1 + \beta_n \phi_n^+ \phi_n^- & 0 \\ -\beta_n (\phi_n^+)^2 & 1 - \beta_n \phi_n^+ \phi_n^- \end{pmatrix} + \begin{pmatrix} 0 & \beta_n (\phi_n^-)^2 \\ 0 & 0 \end{pmatrix} \tag{3.6}$$

$$=: G_n + E_n. \tag{3.7}$$

The advantage of such a decomposition is as follows: let Σ_n be defined recursively as

$$\Sigma_n := \begin{cases} -\beta_n (\phi_n^+)^2 \prod_{j=1}^{n-1} (1 + \beta_j \phi_j^+ \phi_j^-) + (1 - \beta_n \phi_n^+ \phi_n^-) \Sigma_{n-1}, & n \geq 2; \\ -\beta_1 (\phi_1^+)^2, & n = 1, \end{cases} \tag{3.8}$$

and let

$$\Pi_n^\pm = \prod_{j=1}^n (1 \pm \beta_j \phi_j^+ \phi_j^-) := \prod_{j=1}^n p_j^\pm. \tag{3.9}$$

Under such definitions, we have a closed form for the product $G_n G_{n-1} \ldots G_1$:

$$G_n G_{n-1} \cdots G_1 = G_n \begin{pmatrix} \Pi_{n-1}^+ & 0 \\ \Sigma_{n-1} & \Pi_{n-1}^- \end{pmatrix} = \begin{pmatrix} \Pi_n^+ & 0 \\ \Sigma_n & \Pi_n^- \end{pmatrix}. \tag{3.10}$$

While G_n is an approximation for the recurrence matrix $I + M_n$, we shall show that the product $G_n G_{n-1} \cdots G_1$ will serve as an approximation to the actual recurrence relation (2.3) for $a_n^\pm$ under Trench-type conditions on $V_n^0 - V_n$:

Theorem 3.2. *Let $\phi_n^\pm$ be independent solutions to the difference equation $-\Delta + V^0 = 0$. Consider a potential V such that*

$$\sup_n |\Sigma_n| < \infty \text{ and } \sum_n^\infty \left| (V_n - V_n^0)(\phi_n^+ \phi_n^-) \right| < \infty. \tag{3.11}$$

If either $\sum_{n=1}^\infty |\phi_n^-|^2 < \infty$ or $\sum_{n=1}^\infty |V_n^0 - V_n| < \infty$, then any non-trivial solution ψ to the equation $-\Delta + V = 0$ can be written as $\psi_n = a_n^+ \phi_n^+ + a_n^- \phi_n^-$ such that one of the following is true:

1. *There is an integer p such that*

$$a^+_{p+k} = a^+_p \neq 0, \quad a^-_{p+k} = 0 \quad \forall k \geq 0. \tag{3.12}$$

2. *There is an integer p such that*

$$a^-_{p+k} = a^-_p \neq 0, \quad a^+_{p+k} = 0 \quad \forall k \geq 0. \tag{3.13}$$

3. *There exists a constant $f_\infty \neq 0$ such that $a^+_n = \Pi^+_n f_\infty + o(1)$ and $a^-_n = \Sigma_n f_\infty + o(1)$.*

4. *There exist constants f^+_∞, f^-_∞ with $f^-_\infty \neq 0$ such that $a^+_n = \Pi^+_n f^+_\infty + o(1)$ and $a^-_n = \Sigma_n f^+_\infty + f^-_\infty + o(1)$.*

In Example 6.1, we will construct a potential V such that $\sup\limits_n |\Sigma_n| < \infty$ but Σ_n fluctuates as n goes to infinity.

Next, we prove a result such that $|\Sigma_n|$ may go to infinity:

Theorem 3.3. *Let the definitions be the same as in Theorem 3.2 and V be a potential such that*

$$V_n - V^0_n \geq 0 \, (or \ \leq 0) \quad \forall n \in \mathbb{N} \ and \ \sum_n^\infty \left|(V_n - V^0_n)(\phi^+_n \phi^-_n)\right| < \infty. \tag{3.14}$$

Moreover, we require that $\sup_n |(\phi^-_n)^2 \Sigma_n| < \infty$.

If $\lim\limits_{n\to\infty} |\Sigma_n| = \infty$, then exactly one of the following must be true: as $n \to \infty$,

1. *$a^+_n = \Pi^+_n (a^+_\infty + o(1)) \neq 0$ and $a^-_n = \Sigma_{n-1}(a^+_\infty + o(1))$;*

2. *$a^+_n \to 0$ and $a^-_n \to a^-_\infty \neq 0$.*

If $\sup\limits_n |\Sigma_n| < \infty$, the reader may refer to Theorem 3.2.

Proof of Theorem 3.2. Let f^+_{n+1} and f^-_{n+1} be defined implicitly in (3.15) below:

$$\begin{pmatrix} a^+_{n+1} \\ a^-_{n+1} \end{pmatrix} = \begin{pmatrix} \Pi^+_n & 0 \\ \Sigma_n & \Pi^-_n \end{pmatrix} \begin{pmatrix} f^+_{n+1} \\ f^-_{n+1} \end{pmatrix}. \tag{3.15}$$

First, we want to prove that

$$\sup_n |f^+_n| + |f^-_n| < \infty. \tag{3.16}$$

Then we prove that exactly one of the following must be true:

1. $\lim\limits_{n\to\infty} f^+_n =: f^+_\infty \neq 0$ and $\lim\limits_{n\to\infty} f^-_n = 0$.

2. $\lim\limits_{n\to\infty} f^+_n =: f^+_\infty$ exists and $\lim\limits_{n\to\infty} f^-_n \neq 0$.

To begin, we observe that

$$f^+_{n+1} - f^+_n = \frac{a^+_{n+1}}{\Pi^+_n} - \frac{a^+_n}{\Pi^+_{n-1}} = \frac{a^+_{n+1} - p^+_n a^+_n}{\Pi^+_n} = \frac{\beta_n (\phi^-_n)^2 a^-_n}{\Pi^+_n} = \frac{\beta_n (\phi^-_n)^2 (\Sigma_{n-1} f^+_n + \Pi^-_{n-1} f^-_n)}{\Pi^+_n}, \tag{3.17}$$

which, by the triangle inequality, implies that

$$|f_{n+1}^+| \le |f_n^+| \left(1 + \frac{\beta_n(\phi_n^-)^2\Sigma_{n-1}}{\Pi_n^+}\right) + |f_n^-| \left|\frac{\beta_n(\phi_n^-)^2\Pi_{n-1}^-}{\Pi_n^+}\right|. \tag{3.18}$$

$$f_{n+1}^- - f_n^- = \frac{a_{n+1}^- - \Sigma_n f_{n+1}^+}{\Pi_n^-} - \frac{a_n^- - \Sigma_{n-1} f_n^+}{\Pi_{n-1}^-} = \frac{a_{n+1}^- - p_n^- a_n^- - \Sigma_n f_{n+1}^+ + p_n^- \Sigma_{n-1} f_n^+}{\Pi_n^-}. \tag{3.19}$$

By (2.3),

$$a_{n+1}^- - p_n^- a_n^- = -\beta_n(\phi_n^+)^2 a_n^+ = -\beta_n(\phi_n^+)^2 \Pi_{n-1}^+ f_n^+ = (\Sigma_n - p_n^- \Sigma_{n-1}) f_n^+. \tag{3.20}$$

Thus, (3.19) becomes

$$f_{n+1}^- - f_n^- = \frac{\Sigma_n(f_n^+ - f_{n+1}^+)}{\Pi_n^-} = \frac{\beta_n(\phi_n^-)^2\Sigma_n(\Sigma_{n-1}f_n^+ + \Pi_{n-1}^- f_n^-)}{\Pi_n^-}, \tag{3.21}$$

which implies that

$$|f_{n+1}^-| \le \left|f_n^-\right| \left(1 + \left|\frac{\beta_n(\phi_n^-)^2\Sigma_n}{p_n^-}\right|\right) + |f_n^+| \left|\frac{\beta_n(\phi_n^-)^2\Sigma_n\Sigma_{n-1}}{\Pi_n^-}\right|. \tag{3.22}$$

Add (3.18) to (3.22). Since $\sum_{n=1}^\infty |\beta_n\phi_n^+\phi_n^-| < \infty$, $\Pi_n^\pm$ converges to a non-zero limit and $p_n^\pm \to 1$. Moreover, $\sup_n |\Sigma_n| < \infty$, so there is a constant K such that

$$|f_{n+1}^+| + |f_{n+1}^-| \le (1 + K|\beta_n(\phi_n^-)^2|)\left(|f_n^+| + |f_n^-|\right), \tag{3.23}$$

which implies (3.16).

Dichotomy 3.1. There are only two mutually exclusive possibilities for f_n^+ and f_n^-:

1. For any pair consisting of an integer N and a constant $M > 0$, there exists an integer $p = p(N, M) > N$ such that
$$M\left|f_p^+\right| > |f_p^-|. \tag{3.24}$$
2. There exist an integer N_0 and a constant M_0 such that
$$\left|f_n^+\right| \le M_0|f_n^-| \quad \forall n \ge N_0. \tag{3.25}$$

Suppose we are in Case 1. Note that (3.24) implies that $f_p^+ \neq 0$, because if $f_p^+ = 0$, then $f_p^- = 0$ which implies $a_p^+ = a_p^- = 0$, and hence $a_n^\pm \equiv 0$. This contradicts the assumption that ψ is a non-trivial solution.

Let p be the integer given in (3.24). We shall specify the choice of N and M later in the proof. Let

$$r_n = \frac{f_n^-}{f_n^+}. \tag{3.26}$$

Note that by the triangle inequality and (3.17),

$$\left|\frac{f_{p+1}^+}{f_p^+}\right| \ge 1 - \left|\frac{f_{p+1}^+ - f_p^+}{f_p^+}\right| \ge 1 - K_1|\beta_p(\phi_p^-)^2|(1 + |r_p|) > 0. \tag{3.27}$$

Thus, $f_p^+ \neq 0$ implies $f_{p+1}^+ \neq 0$. Furthermore, by inverting (3.27), we obtain

$$\left|\frac{f_p^+}{f_{p+1}^+}\right| \leq \frac{1}{1 - K_1|\beta_p(\phi_p^-)^2(1+|r_p|)|} < K_2. \tag{3.28}$$

Clearly, both r_p and r_{p+1} are well defined as $f_p^+, f_{p+1}^+ \neq 0$. Observe that by the triangle inequality,

$$|r_{p+1} - r_p| \leq \left|\frac{f_{p+1}^- - f_p^-}{f_p^+}\right| \left|\frac{f_p^+}{f_{p+1}^+}\right| + \left|\frac{f_{p+1}^+ - f_p^+}{f_p^+}\right| \left|\frac{f_p^-}{f_p^+}\right| \left|\frac{f_p^+}{f_{p+1}^+}\right|. \tag{3.29}$$

By by (3.21),

$$\left|\frac{f_{p+1}^- - f_p^-}{f_p^+}\right| \leq \left|\frac{\beta_p(\phi_p^-)^2\Sigma_p}{\Pi_p^+}\right| \Big(|\Sigma_{p-1}| + |r_p||\Pi_{p-1}^-|\Big) \leq (1+|r_p|)K_3|\beta_p(\phi_p^-)^2|. \tag{3.30}$$

Similarly, by (3.17),

$$\left|\frac{f_{p+1}^+ - f_p^+}{f_p^+}\right| \leq \left|\frac{\beta_p(\phi_p^-)^2}{\Pi_p^+}\right| \Big(|\Sigma_{p-1}| + |r_p||\Pi_{p-1}^-|\Big) \leq K_4|\beta_p(\phi_p^-)^2|(1+|r_p|). \tag{3.31}$$

By the triangle inequality,

$$\begin{aligned} |r_{p+1}| \leq |r_{p+1} - r_p| + |r_p| \leq |r_p| \left[1 + \frac{K_5(1+|r_p|)|\beta_p(\phi_p^-)^2|}{1 - K_1|\beta_p(\phi_p^-)^2(1+|r_p|)|}\right] + \frac{K_5|\beta_p(\phi_p^-)^2|}{1 - K_1|\beta_p(\phi_p^-)^2(1+|r_p|)|} \\ = |r_p| + \frac{(1+|r_p|+|r_p|^2)K_5|\beta_p(\phi_p^-)^2|}{1 - K_1|\beta_p(\phi_p^-)^2(1+|r_p|)|}. \end{aligned} \tag{3.32}$$

In particular, if $|r_p| < 1$, then $1 + |r_p| + |r_p|^2 \leq 1 + 2|r_p|$. Besides, when p is large (which will be the case),

$$1 - K_1|\beta_p(\phi_p^-)^2(1+|r_p|)| > 1 - 2K_1|\beta_p(\phi_p^-)^2| > 1/2. \tag{3.33}$$

Hence, (3.32) becomes

$$|r_{p+1}| \leq |r_p| + (1+2|r_p|)K_6|\beta_p(\phi_p^-)^2| = |r_p|\left(1 + 2K_6|\beta_p(\phi_p^-)^2|\right) + K_6|\beta_p(\phi_p^-)^2|. \tag{3.34}$$

Hence, by an inductive argument we can prove that

$$|r_{p+k}| \leq \left(\prod_{j=0}^{k-1}(1+\eta_{p+j})\right)\left(|r_p| + \sum_{j=0}^{k-1}\eta_{p+j}\right), \tag{3.35}$$

where

$$\eta_k := 2K_6|\beta_k(\phi_k^-)^2| \geq 0. \tag{3.36}$$

Since either β_k or $|\phi_k^-|^2$ are summable, $\eta_k \in \ell^1$. Hence,

$$P_p := \prod_{j=0}^{\infty}(1+\eta_{p+j}) < \prod_{j=0}^{\infty}(1+\eta_j) := P_\infty < \infty \tag{3.37}$$

and

$$S_p := \sum_{j=0}^{k-1} \eta_{p+j} < K_7 \sum_{j=0}^{\infty} \eta_{p+j} \to 0 \text{ as } p \to \infty. \tag{3.38}$$

Here is how we choose M and N: given $\epsilon > 0$, choose M, N such that

$$|r_p| < M < \frac{\epsilon}{2P_\infty} \text{ and } \sup_{k \geq N} S_k < \frac{\epsilon}{2P_\infty}. \tag{3.39}$$

It is guaranteed that there exists an integer $p > N$ such that $|r_p| < M$. By (3.35),

$$|r_{p+k}| \leq P_\infty \left(|r_p| + S_p\right) < P_\infty \left(\frac{\epsilon}{2P_\infty} + \frac{\epsilon}{2P_\infty}\right) < \epsilon. \tag{3.40}$$

In other words, if (3.24) is true, then $\lim_{n\to\infty} r_n = 0$. Apply this to (3.31), we get

$$\left|\frac{f^+_{n+1}}{f^+_n} - 1\right| \leq K_7 |\beta_n (\phi^-_n)^2| \in \ell^1. \tag{3.41}$$

Since $\log z$ is analytic near $z = 1$, in a neighborhood of 1 there is a constant K_8, arbitrarily close to 1, such that

$$|\log z| = |\log z - \log 1| \leq K_8 |z - 1|. \tag{3.42}$$

Put $z = f^+_{n+1}/f^+_n$. By (3.41),

$$\left|\log \frac{f^+_{n+1}}{f^+_n}\right| = O\left(|\beta_n (\phi^-_n)^2|\right) \in \ell^1. \tag{3.43}$$

Moreover, by the argument following (3.27), we know that there exists an integer p such that $f^+_n \neq 0$ for all $n \geq p$. Therefore,

$$\log f^+_n = \sum_{j=p}^{n-1} \left(\log \frac{f^+_{j+1}}{f^+_j}\right) + \log f^+_p. \tag{3.44}$$

That implies the existence of $\lim_{n\to\infty} \log f^+_n$, and

$$\lim_{n\to\infty} f^+_n := f^+_\infty \neq 0. \tag{3.45}$$

Together with the proven fact that $r_n \to 0$, we obtain

$$\lim_{n\to\infty} f^-_n = 0. \tag{3.46}$$

Next, suppose (3.25) is true. If $f^-_n \equiv 0$, then $f^+_n \equiv 0$ for all $n \geq N_0$, which contradicts with the assumption that ψ is a non-trivial solution.

Now suppose there is an $m \geq N_0$ such that $f^-_m \neq 0$. Divide both sides of (3.21) by f^-_m. Then we obtain:

$$\left|\frac{f^-_{m+1}}{f^-_m} - 1\right| \leq K_9 |\beta_m (\phi^-_m)^2|. \tag{3.47}$$

Using the same argument as in Case 1, we can prove that $f^-_m \neq 0$ implies $f^-_{m+1} \neq 0$, which allows us to apply the same logarithmic argument to prove that

$$\lim_{n\to\infty} f^-_n := f^-_\infty \neq 0. \tag{3.48}$$

By (3.17),

$$f_{n+1}^{+} = \left(1 + \beta_n(\phi_n^-)^2 \frac{\Sigma_{n-1}}{\Pi_n^+}\right) f_n^+ + \left(\beta_n(\phi_n^-)^2 \frac{\Pi_{n-1}^-}{\Pi_n^+}\right) f_n^- = (1 + O(|\beta_n(\phi_n^-)^2|)) f_n^+ + O(|\beta_n(\phi_n^-)^2|). \tag{3.49}$$

Hence,

$$\lim_{n\to\infty} f_n^+ = f_\infty^+ \text{ exists.} \tag{3.50}$$

□

Proof of Theorem 3.3. The proof is very similar to the one of Theorem 3.2. The only difference in the proof is that instead of f_n^- and r_n we consider

$$\hat{f}_n^- := \frac{f_n^-}{\Sigma_{n-2}} \text{ and } \hat{r}_n = \frac{\hat{f}_n^-}{f_n^+}. \tag{3.51}$$

In place of $|r_{n+1} - r_n|$ in (3.29), we replace it with

$$\left|\hat{r}_{n+1} - \frac{\Sigma_{n-2}}{\Sigma_{n-1}} \hat{r}_n\right| \tag{3.52}$$

and make use of the fact that $|\Sigma_{n-1}/\Sigma_n| \leq 1$ for all n. □

4 Construction of Comparison Equations

In this section we turn to the problem of determining the asymptotic behavior of solutions of (1.1) as $n \to \infty$ given a potential V_n, where V_n can be either bounded or unbounded. We shall construct explicit comparison equations with respect to which we can call upon the perturbation results of the earlier sections of this article. The construction will require a discrete replacement for the Liouville-Green (familiarly, WKB) approximation, which is a well-known and quite useful tool for this purpose in the setting of ordinary differential equations [17, 22].

Our ansatz is that given an equation of the type (1.1), a related equation is to be sought for which the solutions are of the form

$$\phi_n^{\pm} = z_n \prod_{\ell=1}^{n} S_\ell^{\pm 1}. \tag{4.1}$$

Recall that in the Liouville-Green approximation to ordinary differential equations of Schrödinger type a comparison is made to a similar equation having a solution basis in the form

$$V(x)^{-1/4} \exp\left(\pm \int V(x)^{1/2} dx\right)$$

[22]. In common with previous authors, we replace the exponential function containing an "action integral" by a product of the quantities we designate S_n, but we innovate with an additional prefactor z_n, to be specified below in (4.8). This is designed to bring simplifications in the discrete case analogous to those resulting from the prefactor $V(x)^{-1/4}$ in the continuous case.

Before we state the main results of this section, we pause to point out a connection between the factor z_n and the Green matrix for the Schrödinger operator $-\Delta + \widetilde{V}$, *viz.*, for $n > m$,

$$G_{nm} = \phi_{\min(m,n)}^+ \phi_{\max(m,n)}^- = z_n z_m \prod_{\ell=m+1}^{n} \frac{1}{S_\ell}, \tag{4.2}$$

for which $(-\Delta + \widetilde{V})G$ is the identity operator, by a direct computation. In case $n = m$,

$$G_{nn} = z_n^2. \tag{4.3}$$

In the following section we study the diagonal elements of the Green matrix and show that they are directly related to the behavior at infinity of solutions and to the notion of an Agmon metric. (cf. [11, Section 4]).

In order to determine z_n, we recall the constancy of the Wronskian of solutions to equations of the type (1.1). To simplify the discussion, we take $W = 1$, which can be arranged by scaling. Given our assumptions it implies that

$$z_n z_{n+1} \left(S_{n+1} - \frac{1}{S_{n+1}} \right) = 1. \tag{4.4}$$

Guided by the case of a constant potential, we expect that if V_n is well-behaved, then a good choice for S_n is one of the solutions of $S_n + S_n^{-1} = V_n + 2$. This turns out to be adequate in some bounded cases, but a more sophisticated choice is necessary when, for example, V_n is allowed to be unbounded. We remark that the choice is not unique, because different choices lead to the same asymptotic behavior if the comparison potentials they lead to are sufficiently close. Our discussion will proceed under the supposition that $V_n > 0$ for large n; the case where $V_n < -4$ for large n is similar with the systematic sign changes mentioned in Remark 1.

To determine the best choices for S_n and z_n, we consider the equation

$$S_n + \frac{1}{S_n} = b_n, \tag{4.5}$$

which is effectively a quadratic, and let S_n be the root of larger magnitude, i.e.,

$$S_n = \frac{b_n + \sqrt{b_n^2 - 4}}{2}, \quad \text{where } |b_n| \geq 2. \tag{4.6}$$

We observe that the relationship

$$S_n - \frac{1}{S_n} = \sqrt{b_n^2 - 4} \tag{4.7}$$

necessarily follows.

To be consistent with the Wronski identity (4.4) we must set

$$z_n := C_z^{(-1)^n} \sqrt{\frac{(b_{n-1}^2 - 4)(b_{n-3}^2 - 4) \cdots}{(b_n^2 - 4)(b_{n-2}^2 - 4) \cdots}} \tag{4.8}$$

for all n, where the constant C_z will be chosen below. (We clarify that the prefactor simply alternates between C_z and its reciprocal, depending on whether n is even or odd.)

The comparison functions $\phi^{\pm}$ both solve a Schrödinger equation with potential $\widetilde{V}_n$ given by

$$\widetilde{V}_n := \frac{\Delta \phi_n^{\pm}}{\phi_n^{\pm}} = \frac{z_{n+1}}{z_n} S_{n+1} + \frac{z_{n-1}}{z_n} \frac{1}{S_n} - 2. \tag{4.9}$$

(This equation is true by direct substitution for ϕ^+; to see that is it also true for ϕ^- requires also substituting from (4.4); cf. a similar argument for (5.5) in §4.) We shall in fact show that there is a choice of ways to choose b_n that will lead to a sufficient convergence rate of $\widetilde{V}_n - V_n$, and that

the the logarithm of the quantity S_n can be regarded as an Agmon metric [2, 18] controlling the behavior of solutions ϕ of (1.1) at infinity.

For clarity, we first consider the case where V_n is bounded and $V_n \geq C > 0$ for all $n \geq N_0$. Without loss of generality we may assume that $N_0 = 1$, because this does not affect the large-n behavior of a solution basis. (This simply allows us to avoid choosing phases for some square-roots of quantities that might otherwise not be positive.)

In the case of bounded, slowly varying potentials V_n, Theorem 4.1 contains estimates for S_n and z_n and uses them to control the solutions and Green matrix of the comparison equation $(-\Delta+\widetilde{V})\phi = 0$. The construction in Theorem 4.1 is guided by the special case of a constant potential.

In Theorem 4.2 that follows, we shall present a more general result which covers potentials that are convergent to a finite limit under more relaxed assumptions on V_n.

Finally, in Theorem 4.3 we show that the method proposed in this section also works for unbounded potentials that possibly fluctuate.

Theorem 4.1. *(bounded and slowly varying potential) Suppose that for some $C > 0$, $C \leq V_n \in \ell^\infty$, and that $n(V_{n+1} - V_n) \in \ell^1$. Choose*

$$b_n = b_n^{bdd} := V_n + 2. \tag{4.10}$$

This implies (with a short calculation) that

$$S_n = S_n^{bdd} := \frac{1}{2}\left(V_n + 2 + \sqrt{V_n(V_n+4)}\right). \tag{4.11}$$

The factor z_n is determined by (4.8). Then

(a)

$$S_{n+1}^{bdd} - S_n^{bdd} \in \ell^1. \tag{4.12}$$

(b) V_n converges to a nonzero limit V_∞ as $n \to \infty$, and

$$C_z := (V_\infty(V_\infty+4))^{-1/4} \prod_{m=1}^{\infty} \sqrt{\frac{V_{2m}(V_{2m}+4)}{V_{2m-1}(V_{2m-1}+4)}} \tag{4.13}$$

is well defined through a finite convergent product.

(c) Under this definition of C_z and the one of z_n in (4.8), $z_n z_{n+1} = 1/\sqrt{V_n(V_n+4)}$ and

$$z_m - \frac{1}{[V_\infty(V_\infty+4)]^{1/4}} \in l^1. \tag{4.14}$$

(d) The comparison potential $\widetilde{V}_n$ defined in (4.9), satisfies $\lim_{n\to\infty} \widetilde{V}_n = V_\infty$, and the Green matrix for $-\Delta + \widetilde{V}$ and the product $\phi_n^+ \phi_n^-$ are uniformly bounded.

(e) $\widetilde{V}_n - V_n \in \ell^1$, and therefore, identifying $\widetilde{V}$ with the comparison potential V^0 in (1.2), the solutions of (1.1) and (1.2) are asymptotically equivalent in the sense of Theorem 2.2.

Proof. The proof for (a) is a direct application of Taylor's Theorem: Following (4.6), we consider the function $f(x) = 1/2(x + 2 + \sqrt{x(x+4)})$, which is differentiable for all $x > 0$. In particular, if $x, y > C > 0$, $f(y) = f(x) + R(x, y)$, where $R(x, y) = (y - x)(r + 2)/2\sqrt{r(r+4)}$ for some r between x, y, implying that $R(x, y)$ is uniformly bounded in x, y if $x, y > C > 0$. Since $S_{n+1}^{bdd} = f(V_{n+1})$ we can write

$$S_{n+1}^{bdd} = S_n^{bdd} + R(V_n, V_{n+1}) = S_n^{bdd} + O(|V_{n+1} - V_n|), \tag{4.15}$$

which proves (a).

The fact that if the differences $V_{n+1} - V_n$ are summable, then V_n has a limit is immediate. To establish the convergence of the product (4.13), let $\delta_m := V_m - V_{m-1}$. Then

$$V_m(V_m + 4) = V_{m-1}(V_{m-1} + 4) + \delta_m(2V_{m-1} + \delta_m + 4) \tag{4.16}$$

and since $0 < C \le V_n < V_n + 4$ for all n,

$$\left|\delta_m \frac{2V_{m-1} + \delta_m + 4}{V_{m-1}(V_{m-1} + 4)}\right| = O\left(|\delta_m|\right). \tag{4.17}$$

As a result,

$$\frac{V_m(V_m + 4)}{V_{m-1}(V_{m-1} + 4)} = 1 + O\left(\left|\frac{V_m - V_{m-1}}{V_{m-1}}\right|\right), \tag{4.18}$$

which implies that $\ln \frac{V_m(V_m+4)}{V_{m-1}(V_{m-1}+4)} \in \ell^1$. By taking the logarithm in (4.13), the product therefore converges, and is easily seen to be nonzero. This proves (b).

The same argument for (b) establishes the convergence as $m \to \infty$ of z_{2m} and of z_{2m+1}, separately. The choice of the prefactor in (4.13) ensures that the two limits are the same. The more precise statement (4.14) is where the assumption that not only $V_{n+1} - V_n$ but also $n(V_{n+1} - V_n) \in \ell^1$ is needed. From the definition of z_n it can be seen (by taking logs and using Taylor's theorem) that $|z_{n+2} - z_n|$ is dominated by a constant times $|V_{n+1} - V_n| + |V_n - V_{n-1}|$. Thus $\sum_m |z_m - z_\infty|$ is dominated by a constant times

$$\sum_{m=1}^{\infty} \sum_{k=m}^{\infty} |V_k - V_{k-1}|,$$

which by reversing the order of summation equals

$$\sum_{k=1}^{\infty} (k-1)|V_k - V_{k-1}| < \infty$$

by assumption. The other statements in (c) follow by (4.2) and (4.3).

Finally, by (4.9),

$$R_n := V_n - \widetilde{V}_n = V_n - \left(\frac{z_{n+1} S_{n+1}^{bdd}}{z_n} + \frac{z_{n-1}}{z_n S_n^{bdd}} - 2\right) = (V_n + 2) - \left(\frac{z_{n+1} S_{n+1}^{bdd}}{z_n} + \frac{z_{n-1}}{z_n S_n^{bdd}}\right). \tag{4.19}$$

With the aid of (4.11),

$$R_n = (S_{n+1} - S_n) + \left(\frac{z_{n+1}}{z_n} - 1\right) S_{n+1} + \left(\frac{z_{n-1}}{z_n} - 1\right) \frac{1}{S_n}.$$

Since S_n and z_n both have finite nonzero limits and $S_n - \lim_{k\to\infty} S_k$ and $z_n - z_\infty$ are both ℓ^1, each of these three terms is easily seen to belong to ℓ^1.

$\square$

Remark 3. The quantity C_z is analogous to the exponential of an action integral in the continuous situation, which shows up in "tunneling" effects. We summarize that in the case where $0 < C \le V_n$ and $V_{n+1} - V_n \in \ell^1$, there is a Liouville-Green basis of comparison functions for (1.1), and The perturbation method of §2 lets the solutions $\{\psi_n^\pm\}$ be determined from that basis through an

iteration that converges for all $n \geq N$ for some finite N. To collect the details in one formula, the Liouville-Green basis is of the explicit form

$$\phi_n^{\pm} = \sqrt{\frac{V_{n-1}(V_{n-1}+4)V_{n-3}(V_{n-3}+4)\dots}{V_n(V_n+4)V_{n-2}(V_{n-2}+4)\dots}} \prod_k^n \left(\frac{V_k + 2 + \sqrt{V_k(V_k+4)}}{2} \right)^{\pm 1} \tag{4.20}$$

(dropping the normalization factors C_z or, resp., $1/C_z$).

The next result concerns potentials that are convergent to a finite limit under more relaxed assumptions on V_n.

Theorem 4.2. *(general bounded potential) Let $S_n + 1/S_n = b_n$. where b_n is a bounded function of the potential V such that for all n, $C_1 > b_n > C_2 > 2$ for some constants C_1, C_2. If $\sum_n |b_{n+1} - b_n| < \infty$, then*

(a)

$$S_{n+1} - S_n = O(|b_{n+1} - b_n|).$$

(b)

$$P_M := \prod_{m=1}^{M} \left(\frac{b_{2m}^2 - 4}{b_{2m-1}^2 - 4} \right) \to P_\infty \tag{4.21}$$

is well defined through a finite convergent product. If $b_n \neq b_{n+1}$ for all n, then $P_\infty \neq 0$.

(c) If $V_{n+1} - V_n \in \ell^1$, we can choose b_n in a way such that $b_{n+1} - b_n \in \ell^1$ and that $\widetilde{V}_n - V_n \in \ell^1$. Some appropriate choice will be shown explicitly in the proof below.

Proof of Theorem 4.2. The proof for (a) is a direct application of Taylor's Theorem. Following (4.6), we consider the function $f(x) = 1/2(x + \sqrt{x^2 - 4})$, which is differentiable for all $x^2 > 4$. In particular, if $x, y > 2$, $f(y) = f(x) + (y - x)R(x, y)$, where $R(x, y) = r/2\sqrt{r^2 - 4}$ for some r between x, y, implying that $R(x, y)$ is uniformly bounded in x, y if $x, y > 2$. Since $S_{n+1} = f(b_{n+1})$ we can write

$$S_{n+1} = S_n + (b_{n+1} - b_n)R(b_n, b_{n+1}) = S_n + O(|b_{n+1} - b_n|), \tag{4.22}$$

which proves (a).

Note that (4.30) can be expressed as

$$\begin{aligned} \widetilde{V}_n - V_n = {} & \underbrace{\frac{1}{2} \left[\frac{b_n}{\sqrt{b_n^2 - 4}} - \frac{b_{n+1}}{\sqrt{b_{n+1}^2 - 4}} \right] \frac{(b_n^2 - 4)(b_{n-2}^2 - 4)\cdots}{(b_{n-1}^2 - 4)(b_{n-3}^2 - 4)\cdots}}_{(I)} \\ & + \underbrace{\left[\frac{b_{n+1}}{\sqrt{b_{n+1}^2 - 4}} \right] \frac{(b_n^2 - 4)(b_{n-2}^2 - 4)\cdots}{(b_{n-1}^2 - 4)(b_{n-3}^2 - 4)\cdots} - (V_n + 2)}_{(II)} . \end{aligned} \tag{4.23}$$

Clearly, (I) is summable in n because both S_n and b_n are of bounded variation. Hence, we are left with

$$\text{(II)} = \frac{b_n}{\sqrt{b_n^2 - 4}} \frac{(b_n^2 - 4)(b_{n-2}^2 - 4)\cdots}{(b_{n-1}^2 - 4)(b_{n-3}^2 - 4)\cdots} - (V_n + 2). \tag{4.24}$$

To see what possible choices of b_n that will give us the desired convergence, we let

$$J_n = \frac{(b_n^2-4)(b_{n-2}^2-4)\cdots}{(b_{n-1}^2-4)(b_{n-3}^2-4)\cdots}. \tag{4.25}$$

Then we have

$$b_{n+1}^2 - 4 = J_{n+1}J_n \tag{4.26}$$

which implies that

$$(\mathrm{II}) = J_n\sqrt{\frac{J_{n+1}J_n+4}{J_{n+1}J_n}} - (V_n+2) = \sqrt{1+\frac{J_n-J_{n-1}}{J_{n+1}}}\sqrt{(J_{n+1}-J_n)J_n+J_n^2+4} - (V_n+2). \tag{4.27}$$

Therefore, a natural choice for J_n is

$$J_n^2 + 4 = (V_n+2)^2 \tag{4.28}$$

(or equivalently, $J_n = \sqrt{V_n(V_n+4)}$), then $J_{n+1}-J_n = O(|V_{n+1}-V_n|)$ and by (4.26), $b_{n+1}-b_n$ is also $O(|V_{n+1}-V_n|)$. Under this particular choice of J_n,

$$\sqrt{1+\frac{J_n-J_{n-1}}{J_{n+1}}}\sqrt{(J_{n+1}-J_n)J_n+J_n^2+4} = (1+O(V_{n+1}-V_n))(V_n+2). \tag{4.29}$$

In fact, there are a number of choices of J_n that we can choose from. By (4.23) above, $b_{n+1}-b_n \in \ell^1$ and $J_{n+1}-J_n \in \ell^1$ are are sufficient conditions for $\widetilde{V}_n - V_n \in \ell^1$.

We provide a few examples here for the interested reader:

(a) (geometric mean) Let $J_n^2+4 = (V_{n+1}+2)(V_n+2)$. Clearly, under this choice, $J_n^2+4 = (V_n+2)^2+O(|V_{n+1}-V_n|)$ and by (4.26), $b_{n+1}-b_n = O(|J_{n+1}-J_{n-1}|) = O(|V_{n+1}-V_n|+|V_n-V_{n-1}|) \in \ell^1$.

(b) (arithmetic mean) Let $J_n^2+4 = [(V_n+2)^2+(V_{n-1}+2)^2]/2$.

(c) (skipping some V_n's) For $k \geq 0$, let $J_{2k}^2+4 = J_{2k+1}^2+4 = (V_{2k}+2)^2$. Then for all n, $J_{n+1}-J_n = O(|V_{n+1}-V_{n-1}|) \in \ell^1$. Hence, $b_{k+1}-b_k = O(V_{k+1}-V_{k-2}) \in \ell^1$.

□

We now turn to the case where V_n is not bounded. Here we find that *the Liouville-Green approximation for the unbounded case simply requires replacing* V_n+2 *by the geometric mean of* V_n+2 *and* $V_{n-1}+2$. That is, the canonical choice in the unbounded case is

$$S_n - \frac{1}{S_n} = \sqrt{(V_n+2)(V_{n-1}+2)},$$

which, we remark, is equivalent to V_n+2 when V_n is bounded and slowly varying. The argument establishing the accuracy of the Liouville-Green approximation runs much as in the simpler, more restricted case, but with correspondingly more complicated details. As before, this choice of S_n is convenient but not unique.

In terms of a general b_n and the S_n related to it according to (4.5), the comparison potential (4.9) becomes

$$\widetilde{V}_n - V_n = \left[\frac{S_{n+1}}{\sqrt{b_{n+1}^2-4}} + \frac{1}{S_n\sqrt{b_n^2-4}}\right]\frac{(b_n^2-4)(b_{n-2}^2-4)\cdots}{(b_{n-1}^2-4)(b_{n-3}^2-4)\cdots} - (V_n+2). \tag{4.30}$$

Note that by (4.6) and the fact that $1/S_n = b_n - S_n$,

$$\frac{S_{n+1}}{\sqrt{b_{n+1}^2 - 4}} = \frac{b_{n+1}}{2\sqrt{b_{n+1}^2 - 4}} + \frac{1}{2} \tag{4.31}$$

$$\frac{1}{S_n\sqrt{b_n^2 - 4}} = \frac{b_n - S_n}{\sqrt{b_n^2 - 4}} = \frac{b_n}{2\sqrt{b_n^2 - 4}} - \frac{1}{2}. \tag{4.32}$$

Hence, (4.30) becomes

$$\widetilde{V}_n - V_n = \frac{1}{2}\left[\frac{b_{n+1}}{\sqrt{b_{n+1}^2 - 4}} + \frac{b_n}{\sqrt{b_n^2 - 4}}\right]\frac{(b_n^2 - 4)(b_{n-2}^2 - 4)\cdots}{(b_{n-1}^2 - 4)(b_{n-3}^2 - 4)\cdots} - (V_n + 2). \tag{4.33}$$

It turns out that the convenient choice of C_z in a situation where V_n is unbounded is simply $C_z = 1$. The theorem reads as follows:

Theorem 4.3. *(unbounded potential) Let V be a potential that satisfies*

$$\sum_n \frac{1}{V_n^{1/2}}\left(\frac{1}{V_{n+1}^{3/2}} + \frac{1}{V_{n-1}^{3/2}}\right) < \infty. \tag{4.34}$$

Let $C_z = 1$ and $b_n = S_n + 1/S_n > 0$ be chosen such that

$$S_n - \frac{1}{S_n} = \sqrt{b_n^2 - 4} = \sqrt{(V_n + 2)(V_{n-1} + 2)}. \tag{4.35}$$

Then
(a) $b_n = \sqrt{(V_n + 2)(V_{n-1} + 2) + 4}$, $z_n = \frac{1}{\sqrt{V_n + 2}}$ and

$$S_n = \frac{\sqrt{(V_n + 2)(V_{n-1} + 2)} + \sqrt{4 + (V_n + 2)(V_{n-1} + 2)}}{2} > 1. \tag{4.36}$$

(b) $\widetilde{V}_n - V_n = O\left(\frac{1}{V_{n+1}^{3/2}V_n^{1/2}} + \frac{1}{V_n^{1/2}V_{n-1}^{3/2}}\right) \in \ell^1$.

(c) The Green matrix $G_{m,n}$ is uniformly bounded.

Remark 4. The condition (4.34) is satisfied by unbounded potentials, including some that fluctuate. An example is given §6.

Proof of Theorem (4.3). Given this choice of $b_n^2 - 4$, we have

$$\frac{(b_n^2 - 4)(b_{n-2}^2 - 4)\cdots}{(b_{n-1}^2 - 4)(b_{n-3}^2 - 4)\cdots} = V_n + 2. \tag{4.37}$$

By the definition of z_n in (4.8), this implies (a).

Therefore, by (4.33) above,

$$\widetilde{V}_n - V_n = \frac{V_n + 2}{2}\left[\frac{b_{n+1}}{\sqrt{b_{n+1}^2 - 4}} + \frac{b_n}{\sqrt{b_n^2 - 4}} - 2\right]. \tag{4.38}$$

We apply the relation $\sqrt{a}-\sqrt{b}=(a-b)/(\sqrt{a}+\sqrt{b})$ to

$$\frac{b_{n+1}}{\sqrt{b_{n+1}^2-4}}-1=\frac{b_{n+1}-\sqrt{b_{n+1}^2-4}}{\sqrt{b_{n+1}^2-4}}=\frac{b_{n+1}^2-(b_{n+1}^2-4)}{\sqrt{b_{n+1}^2-4}\left(b_{n+1}+\sqrt{b_{n+1}^2-4}\right)}=\frac{4}{b_{n+1}\sqrt{b_{n+1}^2-4}+b_{n+1}^2-4}$$
$$=\frac{4}{\sqrt{(V_{n+1}+2)(V_n+2)}\,[(V_{n+1}+2)(V_n+2)+4]+\sqrt{(V_{n+1}+2)(V_n+2)}}, \quad (4.39)$$

which is in the order of $O((V_{n+1}+2)^{3/2}(V_n+2)^{3/2})$.

Next, we obtain a similar formula for $b_n/\sqrt{b_n^2-4}$ and show that it is in the order of $O((V_n+2)^{3/2}(V_{n-1}+2)^{3/2})$. Canceling the V_n+2 term in (4.38), we prove (b).

Statement (c) about the Green matrix and the products of comparison solutions then follows from (4.2) and (4.3). □

5 The diagonal of the Green matrix and discrete Agmon distance

We next show that the approximations derived in §4 for the solutions and Green matrix of (1.1) by constructing a comparison (1.2) are related to exact identities for Green matrices. In particular, we offer a discrete version of the discovery of Davies and Harrell in [11], §4, that the diagonal elements G_{nn} of the Green matrix allow the full solution space and full Green matrix G_{mn} to be recovered formulaically. We build on significant earlier steps in this direction by Chernyavskaya and Shuster [7, 8]. As in [11] we furthermore point out connections between the diagonal of the Green matrix and an Agmon distance for (1.1).

First observe that with respect to the diagonal of the Green function, there are critical differences between the continuous Schrödinger equations explored in [11] and the discrete equations considered here. Consider that, due to Remark 1, if

$$G_{mn}=\psi^+_{\min(m,n)}\psi^-_{\max(m,n)} \quad (5.1)$$

is the Green matrix for some potential function V_n, then the same diagonal elements G_{nn} also belong to the Green matrix for an equation of type (1.1) but with potential function $\widetilde{V}_n=-4-V_n$. Thus the uniqueness of the representation of [11] is lost, at least to this extent.

We have seen in §4 that for the comparison equation solved by the pair of functions (4.1), the factor z_n equals the square root of G_{nn}. Meanwhile, if z_n is given, then S_n is determined via (4.4), and consequently (4.1) provides a basis for the solution space of the comparison (1.2), and (4.2) reproduces a full Green matrix for (1.2). Here we demonstrate that these implications do not rely on the construction of a comparison equation, but hold in generality for (1.1).

Hence let G_{mn} be the Green matrix for any equation of the form (1.1), and simply define $z_n:=\sqrt{G_{nn}}$. (If G_{nn} is negative, a canonical choice of phase could be assigned to z_n, but here we primarily consider the case where $G_{nn}\geq 0$.) We then use (4.4) to define S_n for $M+1\leq n\leq N$ *viz.*, choosing the root analogously to (4.6),

$$S_n^{[z]}:=\frac{1+\sqrt{1+4z_n^2z_{n-1}^2}}{2z_nz_{n-1}}. \quad (5.2)$$

Here we caution that this choice of the root of the quadratic equation for $S_n^{[z]}$ will restrict the possible values of V_n in what follows. A pair of functions $\varphi_n^{\pm}$ can now be defined by the ansatz

(4.1), i.e., when expressed in terms of z_n

$$\varphi_n^{\pm} := z_n \prod_{k=m+1}^{n} \left(\frac{1+\sqrt{1+4z_n^2 z_{n-1}^2}}{2z_n z_{n-1}} \right)^{\pm 1}. \tag{5.3}$$

Remarkably, with this definition, both φ^+ and φ^- solve an equation of the form (1.1), where the potential function V_n is determined from z_n via

$$\begin{aligned} V_n^{[z]} := \frac{\Delta \varphi_n^+}{\varphi_n^+} &= \frac{1+\sqrt{1+4z_n^2 z_{n+1}^2}}{2z_n^2} + \frac{2z_{n-1}^2}{1+\sqrt{1+4z_n^2 z_{n-1}^2}} - 2 \\ &= \frac{z_{n+1}}{z_n} S_{n+1}^{[z]} + \frac{z_{n-1}}{z_n S_n^{[z]}} - 2, \end{aligned} \tag{5.4}$$

provided that $V_n > -2$. (Else a different root must be chosen in (5.2).) To see that $\varphi_n^{\pm}$ solve the same discrete Schrödinger equation, let us separately calculate

$$\frac{\Delta \varphi_n^-}{\varphi_n^-} = \frac{z_{n+1}}{z_n S_{n+1}^{[z]}} + \frac{z_{n-1}}{z_n} S_n^{[z]} - 2, \tag{5.5}$$

and note that since $S_n^{[z]}$ has been chosen to satisfy the equivalent of (4.4), the difference between these last two expressions is

$$\frac{1}{z_n^2} - \frac{1}{z_n^2} = 0.$$

This leads to a theorem in the spirit of [11].

Theorem 5.1. *Suppose that* (1.1) *has two independent positive solutions for* $m \le n \le N$*, with* $N \ge M+2$*, and denote the associated Green matrix* G_{mn}*. Since* $G_{nn} > 0$ *for* $m \le n \le N$*, we may define* $z_n := \sqrt{G_{nn}}$*. In terms of* z_n*, define* $S_n^{[z]}$ *and* $\varphi_n^{\pm}$ *according to* (5.2) *and* (5.3)*. Then*

1. $\varphi_n^{\pm}$ *is an independent pair of solutions of* (1.1) *for* $m < n \le N$*.*
2. $G_{nm} = z_n z_m \prod_{\ell=m+1}^{n} \frac{1}{S_\ell^{[z]}}$*,* $M < m < n \le N$*.*
3. *The potential function is determined from* G_{nn} *by a nonlinear difference equation,*

$$\frac{1}{2}\left(\sqrt{1+4G_{n\,n}G_{n+1\,n+1}} + \sqrt{1+4G_{n\,n}G_{n-1\,n-1}} \right) = (2+V_n)G_{n\,n}. \tag{5.6}$$

Remark 5. The assumption that there are two positive solutions is related to the notion of *disconjugacy* in the theory of ordinary differential equations, cf. [17, 1]. If, for example, $V_n > 0$ for $n \ge N_0$, then it is not difficult to show that no solution can change sign more than once, and that therefore the positivity assumption is satisfied for n sufficiently large. As will be seen in the proof, a necessary condition for the assumption is that $V_n > -2$.

Per Remark 1 the positivity assumption can be replaced by the assumption that there are two solutions $\psi_n^{\pm}$ such that $(-1)^n \psi_n^{\pm} > 0$. A sufficient condition for this is that $V_n < -4$ and a necessary condition is that $V_n < -2$.

Proof. The essential calculation was provided in the discussion before the statement of the theorem. Given that the Wronskian of φ^- and φ^+ is 1, these two functions are linearly independent and therefore a basis for the solution space of

$$(-\Delta + V_n^{[z]})\varphi = 0.$$

Moreover,

$$G_{mn} = \varphi^+_{\min(m,n)}\varphi^-_{\max(m,n)}$$

is a Green function for $-\Delta + V_n^{[z]}$. The crux of the proof is to show that $V_n^{[z]}$ is the same as the original V_n of (1.1).

Because $S_n^{[z]}$ was defined such that

$$S_n^{[z]} - \frac{1}{S_n^{[z]}} = \frac{1}{z_n z_{n-1}}.$$

we may rewrite (5.4) as

$$2 + V_n^{[z]} = \frac{1}{2z_n^2}\left(\sqrt{1+4z_n^2 z_{n+1}^2} + \sqrt{1+4z_n^2 z_{n-1}^2}\right) \tag{5.7}$$

From the definition of z_n and the assumptions of the theorem, we know that for some independent set of positive solutions $\psi_n^\pm$ of (1.1), with Wronskian 1, $z_n^2 = \psi_n^+\psi_n^-$. Therefore

$$\begin{aligned}
4z_n^2 z_{n\pm1}^2 &= 4(\psi_n^+\psi_{n\pm1}^-)(\psi_n^-\psi_{n\pm1}^+)\\
&= (\psi_n^+\psi_{n\pm1}^- + \psi_n^-\psi_{n\pm1}^+)^2 - (\psi_n^+\psi_{n\pm1}^- - \psi_n^-\psi_{n\pm1}^+)^2\\
&= (\psi_n^+\psi_{n\pm1}^- + \psi_n^-\psi_{n\pm1}^+)^2 - 1.
\end{aligned}$$

Hence (5.7) yields

$$\begin{aligned}
2 + V_n^{[z]} &= \frac{1}{2\psi_n^+\psi_n^-}\left(\psi_n^+\psi_{n+1}^- + \psi_n^-\psi_{n+1}^+ + \psi_n^+\psi_{n-1}^- + \psi_n^-\psi_{n-1}^+\right)\\
&= \frac{1}{2\psi_n^+\psi_n^-}\left(\psi_n^+ V_n\psi_n^- + \psi_n^- V_n\psi_n^+\right)\\
&= 2 + V_n,
\end{aligned}$$

as claimed, and establishes (5.6), according to (5.7). □

Formula (5.3) suggests that S_n can be related to an Agmon distance [2, 18], that is, a metric $d_A(m,n)$ on the positive integer lattice such that every ℓ^2 solution ϕ^- of (1.1) satisfies a bound of the form

$$e^{d_A(0,n)}\phi_n^- \in \ell^\infty,$$

and that as a consequence ϕ_n^- decays rapidly as $n \to \infty$. Thus if z_n is bounded we expect an Agmon distance to be something like $\sum_{\ell=m+1}^n \ln S_\ell^{[z]}$. (We write the Agmon distance in this way because a metric on the integer lattice must be in the form of a sum, as the triangle inequality is an equality.) In Agmon's theory, however, the distance function should be a quantity that can be calculated directly from the potential alone, and indeed, the estimates in §4 already imply some bounds of this form. As we shall now see, understanding the diagonal of the Green matrix allows the derivation of Agmonish bounds without the need to control expressions involving $V_{n+1} - V_n$, as in §4. We begin by showing that G_{nn} is comparable to $(V_n + 2)^{-1}$ in a precise sense.

Lemma 5.2. *Suppose that* $\liminf_{n\to\infty} V_n > C > 0$ *and let* G_{mn} *be any Green matrix for* (1.1). *Define*

$$K_A := \sqrt{1 + \left(\frac{2}{C(C+2)}\right)^2} + \frac{2}{C(C+2)}.$$

Then for n *sufficiently large,*

$$\frac{1}{V_n + 2} \le G_{nn} \le \frac{K_A}{V_n + 2}. \tag{5.8}$$

Consequently,

$$\frac{\sqrt{(V_n+2)(V_{n-1}+2)} + \sqrt{4 + (V_n+2)(V_{n-1}+2)}}{2K_A} \le S_n^{[z]} \le \frac{\sqrt{(V_n+2)(V_{n-1}+2)} + \sqrt{4 + (V_n+2)(V_{n-1}+2)}}{2}. \tag{5.9}$$

Remark 6. Note that the upper bound is of the same form as was found for the Liouville-Green approximation in (4.36). For a simpler bound K_A could be replaced in these inequalities by

$$\sqrt{1 + \frac{4}{C^2}} > K_A$$

(see proof).

Proof. The lower bound on G_{nn} is immediate from Statement (3) of Theorem 5.1.

The upper bound in (5.8) requires a spectral estimate. The Green matrix G_{mn} is the kernel of the resolvent operator of a self-adjoint realization of $-\Delta + V$ on $\ell^2([N, \infty))$ for some N, where the boundary condition at $n = N, N+1$ is that satisfied by φ_n^+. Since $-\Delta > 0$ on this space (as an operator), $\inf \mathrm{sp}(-\Delta + V) > C$, and hence, by the spectral mapping theorem, $\|(-\Delta + V)^{-1}\|_{\mathrm{op}} < C^{-1}$. Since $G_{nn} = \langle e_n, (-\Delta + V)^{-1} e_n \rangle$, where $\{e_n\}$ designate the standard unit vectors in ℓ^2, it follows that $G_{nn} < C^{-1}$. Inserting this into (5.4) would already imply (5.8) with K_A replaced by $\sqrt{1 + 4/C^2}$. To improve the constant, replace only the terms $G_{n\pm1\, n\pm1}$ in (5.4) by $1/C$, getting

$$(2 + V_n) \le \frac{\sqrt{1 + \frac{4G_{nn}}{C}}}{G_{nn}}. \tag{5.10}$$

Since

$$\frac{\sqrt{1 + xy}}{x}$$

is a decreasing function of x when $x, y > 0$, an upper bound on G_{nn} is the larger root of (5.10) (which is effectively a quadratic). The claimed upper bound with the constant K_A results by keeping one factor $V_n + 2$ in the solution of the quadratic, replacing the others by $C + 2$.

The bounds on $S_n^{[z]}$ result from inserting the bounds on G_{nn} into (5.2) and collecting terms. □

We can now state some Agmonish bounds.

Corollary 5.3. *Suppose that* $\liminf_{n\to\infty} V_n > C > 0$ *and fix a positive integer* m. *Then the subdominant solution* φ^- *of* (1.1) *satisfies*

(a)

$$\left(\prod_{\ell=m}^{n} \frac{V_\ell + 2}{K_A}\right) \varphi_n^- \in \ell^\infty.$$

(b) If, in addition, $V_{n+1} - V_n \in \ell^1$, then

$$\left(\prod_{\ell=m}^{n} \frac{V_\ell + 2 + \sqrt{V_\ell(V_\ell+4)}}{2}\right) \varphi_n^- \in \ell^\infty.$$

Proof. Recall the representation (5.3). Because z_n is bounded, so is

$$\left(\prod_{\ell}^{n} S_\ell^{[z]}\right) \varphi_n^-.$$

We then use the lower bound on $S_\ell^{[z]}$ from the lemma, but simplify by dropping the 4, which allows the product to telescope in a pleasing way, producing (a).

For (b) we note that the additional assumption on V_n allows us to conclude that φ is well-approximated by the Liouville-Green expression in §4. Since z_n is again bounded, so is

$$\left(\prod_{\ell}^{n} S_\ell\right) \varphi_n^-,$$

using the ansatz (4.1). Finally, we recall (4.6). □

Thus when $\liminf_{n\to\infty} V_n > 0$, a suitable Agmon distance $d_A(m,n)$ for (1.1) is given by

$$\sum_{\ell=m+1}^{n} (\ln(V_l + 2) - \ln K_A),$$

or by

$$\sum_{\ell=m+1}^{n} \ln \frac{V_\ell + 2 + \sqrt{V_\ell(V_\ell+4)}}{2},$$

provided that $\frac{V_{n+1}-V_n}{V_n} \in \ell^1$. The latter can be weakened to the simpler expression

$$\sum_{\ell=m+1}^{n} \ln\left(V_\ell + 1\right).$$

6 Some illustrative examples

In Theorem 3.2, we consider the problem when $\sup_n |\Sigma_n| < \infty$. Here we construct a potential V such that the boundedness condition of Σ_n is satisfied but Σ_n fluctuates as $n \to \infty$:

Example 6.1 (bounded but fluctuating Σ_n)**.** Let $V_n^0 \equiv V$ such that $V \notin [-4, 0]$. Then we may find a non-zero $x \in (-1, 1)$ such that

$$x + \frac{1}{x} = (2 + V). \tag{6.1}$$

The solutions to $-\Delta + V^0 = 0$ are given by

$$\phi_n^- = x^n \text{ and } \phi_n^+ = x^{-n}. \tag{6.2}$$

and the Wronskian W is $x^{-1} - x$. Consider an asymptotically constant potential:

$$V_n^\alpha = V + (-1)^n W x^{2n}. \tag{6.3}$$

In other words, $\beta_n = \beta_n \phi_n^+ \phi_n^- = (-1)^n x^{2n}$ is summable and $\beta_n(\phi_n^+)^2 = (-1)^n$. Therefore, $0 < \sup_n |\prod_{j=1}^n (1 \pm \beta_j \phi_j^+ \phi_j^-)| < \infty$. For $n \geq 1$,

$$\Sigma_n = (-1)^{n+1} \prod_{j=1}^{n-1} (1 + (-1)^j x^{2j}) + (1 - (-1)^n x^{2n}) \Sigma_{n-1} \tag{6.4}$$

with

$$\Sigma_1 = 1, \quad \Sigma_2 = x^2 - x^4, \quad \Sigma_3 = 1 - x^6 + x^8 - x^{10}. \tag{6.5}$$

Using (6.4), it is easy to prove that for $k \in \mathbb{N}$,

$$\lim_{k\to\infty} \Sigma_{2k} = 0 \text{ and } \lim_{k\to\infty} \Sigma_{2k+1} = 1. \tag{6.6}$$

Example 6.2. The main situation we have treated is where $V_n \to V_\infty \notin [-4, 0]$, with $V_n - V_\infty \in \ell^2$, for which the solutions are of exponential type, with a subdominant solution. As a second case, let us suppose that $V_n \to \infty$

Eigenfunctions that decay only polynomially are possible when $V_\infty = 0$ or -4. Suppose, for example, that $\phi_n^- = n^{-\alpha}$ for some $\alpha > 0$. This is a solution to a discrete Schrödinger equation with a potential satisfying

$$V_k = \frac{(\Delta\phi^-)_k}{\phi_k} = -2 + \left(\frac{k}{k+1}\right)^\alpha + \left(\frac{k}{k-1}\right)^\alpha. \tag{6.7}$$

Using a Taylor expansion, we find that

$$V_k = \frac{(\Delta\phi^-)_k}{\phi_k} = \frac{\alpha(\alpha+1)}{k^2} + 0(k^{-4}). \tag{6.8}$$

Thus polynomial decay can be anticipated when the potential decreases like γk^{-2}.

Corollary 6.1. *Suppose that for some $\gamma > 0$,*

$$V_k = \frac{\gamma}{k^2} + W_k,$$

where $kW_k \in \ell^1$. Then equation (1.1) *has a subdominant solution ψ_k^- such that*

$$\lim_{k\to\infty} k^{\frac{1}{2}(1+\sqrt{1+4\gamma})} \psi_k^- = 1. \tag{6.9}$$

For any solution ψ_k that is linearly independent of ψ_k^-,

$$k^{\frac{1}{2}(1-\sqrt{1+4\gamma})} \psi_k \tag{6.10}$$

converges to a finite, nonzero value.

Next, we provide an example such that $|\phi_n^+ \phi_n^-|$ is not bounded, yet the quantity $J_n \in \ell^1$ and therefore Theorems 3.2 and 3.3 apply.

Example 6.3 (sparse perturbation)**.** Consider a potential V^0 such that

$$V^0_n := \begin{cases} \dfrac{2}{(n+1)(n-1)} & n > 1; \\ -\frac{3}{2} & n = 1. \end{cases} \tag{6.11}$$

It is easy to verify that $\phi^-_n := 1/n$ is a solution to the equation $-\Delta + V^0 = 0$ under the convention that $\phi^-_{-1} = 0$. By Corollary 2.8, ϕ^+_n obeys $|\phi^+_n \phi^-_n| \sim Cn$ for some constant C.

Consider a potential V which is a sparse perturbation of V^0:

$$V_n := \begin{cases} V + \dfrac{W}{n^2} & n = 2^k \text{ for some } k \in \mathbb{N}; \\ V & \text{otherwise.} \end{cases} \tag{6.12}$$

Under such definitions, $\beta_n \phi^+_n \phi^-_n \sim C/n$ is sparsely distributed at powers of 2 and hence summable.

Finally, we provide an example for which Liouville-Green approximation is accurate, while the potential fluctuates and diverges as $n \to \infty$.

Example 6.4. Let V^a be defined such that

$$V^a_n = \begin{cases} n^a & \text{if n is odd;} \\ 1 & \text{if n is oven,} \end{cases} \quad a > 2. \tag{6.13}$$

Then

$$\sum_n \frac{1}{V_n^{1/2}} \left(\frac{1}{V_{n+1}^{3/2}} + \frac{1}{V_{n-1}^{3/2}} \right) = \sum_{n \text{ is odd}} \frac{2}{n^{a/2}} + \sum_{n \text{ is even}} \left(\frac{1}{(n+1)^{3a/2}} + \frac{1}{(n-1)^{3a/2}} \right) < \infty. \tag{6.14}$$

Appendix: Second-order difference equations and orthogonal polynomials

In this section, we will show how the discrete Schrödinger operator relates to orthogonal polynomials on the real line. We begin by recalling some standard facts; the reader may refer to [25, 26] for a comprehensive introduction to the subject.

Let μ be a non-trivial measure on $\mathbb{R}$ such that for all $n \in \mathbb{N}$, the moments are finite. In other words,

$$\int_{\mathbb{R}} |x|^n d\mu(x) < \infty. \tag{6.15}$$

We form an inner product and a norm on $L^2(\mathbb{R}, d\mu)$ as follows: for any $f, g \in L^2(\mathbb{R}, d\mu)$, we define an inner product and a norm as follows:

$$\langle f, g \rangle = \int_{\mathbb{R}} \overline{f(x)} g(x) d\mu(x), \quad \|f\|^2 = \int_{\mathbb{R}} f(x)^2 d\mu(x). \tag{6.16}$$

By the Gram–Schmidt process, we can orthogonalize $1, x, x^2, \dots$ and obtain the family of monic orthogonal polynomial on the real line with respect to the measure μ, which we denote as $(P_n(x))_{n=0}^{\infty}$. For example, if $\mu = \sqrt{2\pi}^{-1} e^{x^2/2}$, then we obtain the Hermite polynomials; and if $\mu = \chi_{[-1,1]} dx$, then we obtain the Legendre polynomials.

Let $(p_n(x))_{n=0}^{\infty}$ denote the family of normalized orthogonal polynomials, i.e., $\|p_n\|_{\mu}^2 = 1$. It is well-known that the monic and the normalized orthonormal polynomials on the real line satisfy the following recurrence relations

$$xP_n(x) = \qquad P_{n+1}(x) + b_{n+1}P_n(x) + a_n^2 P_{n-1}(x), \tag{6.17}$$

$$xp_n(x) = \qquad a_{n+1}(x)p_{n+1}(x) + b_{n+1}p_n(x) + a_n p_{n-1}(x). \tag{6.18}$$

Note that (6.18) above can be expressed as follows:

$$\begin{pmatrix} b_1 & a_1 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & \dots \\ 0 & a_2 & b_3 & a_3 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} 1 \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix} = x \begin{pmatrix} 1 \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix}. \tag{6.19}$$

The tridiagonal matrix in (6.19) above is called the **Jacobi matrix**. The recurrence relation (6.18) can also be expressed in terms of the 2×2 transfer matrix $A_{n+1}(x)$ as follows

$$\begin{pmatrix} p_{n+1}(x) \\ a_{n+1}p_n(x) \end{pmatrix} = \underbrace{{a_{n+1}}^{-1} \begin{pmatrix} x - b_{n+1} & -1 \\ a_{n+1}^2 & 0 \end{pmatrix}}_{A_{n+1}(x)} \begin{pmatrix} p_n(x) \\ a_n p_{n-1}(x) \end{pmatrix}, n \geq 0. \tag{6.20}$$

Observe that the discrete Schrödinger operator with potential V and energy E on f can be written as

$$-f_{n+1} - f_{n-1} + (V_n + 2)f_n = Ef_n. \tag{6.21}$$

Compare (6.21) with (6.18). Note that the discrete Schrödinger equation (6.21) can be seen as having $a_n \equiv 1$ and $b_{n+1} = V_n + 2$ and $X = E$. Hence, orthogonal polynomials associated with the measure with recurrence relations $a_n \equiv 1$ and $b_{n+1} = V_n + 2$ evaluated at $x = E$ can be seen as a solution of the difference equation (6.21) with initial condition $(p_0(x), a_0 p_{-1}(x)) = (1, 0)$.

The solution to (6.20) with initial condition $(0, -1)$ (i.e., $n = 0$) are known as orthogonal polynomials of the second kind, $(q_n(x))_{n=0}^{\infty}$, where $q_n(x)$ is a polynomial of degree $n - 1$. Therefore, $(p_n(x))_{n=0}^{\infty}$ and $(q_n(x))_{n=0}^{\infty}$ form a basis for the solution space of the difference equation (6.21).

However, for the Schrödinger equation, we impose the condition that the solution is square summable (i.e. in $\ell^2(\mathbb{N})$), a property that is not necessarily satisfied by $p_n(E)$. In fact, for any $x_0 \in \mathbb{R}$,

$$\left(\sum_{k=0}^{\infty} p_k(x_0)^2 \right)^{-1} = \mu(x_0). \tag{6.22}$$

Hence, $(p_n(E))_n$ is a solution if and only if E is a pure point μ.

Recall the second-order difference equation (1.13) studied by Geronimo–Smith [13] which was briefly discussed in Section 1. Note that (1.13) can be written in terms of a transfer matrix

$$\begin{pmatrix} y(n+1) \\ y(n) \end{pmatrix} = d(n+1)^{-1} \begin{pmatrix} q(n) & -1 \\ d(n+1) & 0 \end{pmatrix} \begin{pmatrix} y(n) \\ y(n-1) \end{pmatrix} \tag{6.23}$$

which resembles the transfer matrix $A_{n+1}(x)$ in (6.20). Hence, techniques developed to study the asymptotic behavior of orthogonal polynomials can be applied to study ratio asymptotics of the solutions, which determines whether the limit $\lim_{n\to\infty} y(n+1)/y(n)$ exists and what the limit is in the case that it does. For (1.13) and given that $y(n) = \prod_{j=n_0}^{n} u(j)$, ratio asymptotics means

$$\frac{y(n+1)}{y(n)} = \frac{u(n+1)}{u(n)}, \tag{6.24}$$

which explains why it was reasonable for Geronimo–Smith to assume that $\lim_{n\to\infty} u(n+1)/u(n)$ exists should the convergence rates of $q(n)$ and $d(n)$ be sufficiently fast.

For the asymptotic analysis of $p_n(x)$ by means of the transfer matrix when the coefficients are asymptotically identical (meaning $a_n \to a$, $b_n \to b$), the reader may refer to [32].

Acknowledgments

We are grateful to Jeff Geronimo, Nina Chernyavskaya, and Dale Smith for references and helpful remarks.

The second author is supported by the National Science Foundation through nsf-dms 0456611, M. Lacey, P.I.

References

[1] R. P. Agarwal, Difference equations and inequalities, theory, methods and applications, Second Edition, Revised and Expanded, Marcel Dekker, New York, 2000, pp. 998.

[2] S. Agmon, Lectures on exponential decay, Princeton Univ. Press, Mathematical Notes 29. Princeton, 1982.

[3] Z. Benzaid and D. A. Lutz, *Asymptotic representation of solutions of perturbed systems of linear difference equations*, Stud. Appl. Math. **77** (1987) 195–221.

[4] G. D. Birkhoff, General theory of linear difference equations, Trans. Amer. Math. Soc. **12** (1910) 243–284.

[5] P. A. Braun, *WKB method for three-term recursion relations and quasienergies of an anharmonic oscillator*, Teoret. Mat. Fiz. **37** (1978) 355–370, translated as Theoret. and Math. Phys. **37** (1978) 1070–1081.

[6] N. A. Chernyavskaya, *Stability and absolute stability of a three-point difference scheme*, Computers and Math. Appl. **45** (2003)1181–1194.

[7] N. A. Chernyavskaya and L. Shuster, *Regularity of the inversion problem for the Sturm-Liouville difference equation, I. Representation of the Davies-Harrell type for the Green difference function.* J. Math. Analysis Appl. **254** (2000) 150–163.

[8] N. A. Chernyavskaya and L. Shuster, *Regularity of the inversion problem for the Sturm-Liouville difference equation, II. Two-sided estimates for the diagonal of the Green function.* J. Math. Analysis Appl. **254** (2000) 150–163.

[9] N. A. Chernyavskaya and L. Shuster, *Necessary and sufficient conditions for solvability of the Hartman–Wintner problem for difference equations.* J. Difference Eqns. Appl. **14** (2008) 1215–1254.

[10] C. V. Coffman, *Asymptotic Behavior of Solutions of Ordinary Difference Equations*, Trans. Amer. Math. Soc. **110** (1964) 22–51.

[11] E. B. Davies and E. M Harrell II, *Conformally flat Riemannian metrics, Schrödinger operators, and semiclassical approximation*, J. Diff. Eqs. **66** (1987) 165–188.

[12] M. S. P. Eastham, The asymptotic solution of linear differential systems. Applications of the Levinson Theorem. Oxford: Clarendon Press, 1989.

[13] J. S. Geronimo and D. T. Smith, *WKB (Liouville-Green) Analysis of Second Order Difference Equations and Applications*, J. Approx. Theory **69** (1992) 269–301.

[14] J. S. Geronimo and W. Van Assche, *Relative asymptotics for orthogonal polynomials with unbounded recurrence coefficients*, J. Approx. Theory **62** (1990) 47–69.

[15] F. Gesztesy and B. Simon, *The Xi Function*, Acta Math. **176** (1994) 49–71.

[16] E. M. Harrell II and B. Simon, The Mathematical Theory of Resonances whose Widths are Exponentially Small, Duke Math. J. **47** (1980) 845–902.

[17] P. Hartman, Ordinary Differential Equations, Hartman, Baltimore, 1973.

[18] P. Hislop and I. M. Sigal, Introduction to spectral theory, with applications to Schrödinger operators, Springer Applied Mathematical Sciences 113. New York: Springer-Verlag, 1996.

[19] M.G. Krein, On certain new studies in the perturbation theory for self-adjoint operators, in M.G. Krein, Topics in differential and integral equations and operator theory, I. Gohberg, ed. Birkhuser, Basel, 1983, 107–172.

[20] N. Levinson, *The asymptotic nature of solutions of linear systems of differential equations*, Duke Math. J. **15** (1948) 111–126.

[21] A. Máté, P. Nevai, V. Totik, *Asymptotics for orthogonal polynomials defined on a recurrence relation*, Constr. Approx. **1** (1985) 231–248.

[22] F. W. J. Olver, *Asymptotics and Special Functions*, New York: Academic, 1974.

[23] O. Perron, *Über Stabilität und asymptotisches Verhalten der Lösungen eines Systems endlicher Differenzengleichungen*, J. reine angew. Math. (Crelles Journal) **161** (1929) 41–64;

[24] H. Poincaré, *Sur les équations linéaires aux differentielles ordinaires et aux differences finies*, Amer. J. Math. **7** (1885) 203–258.

[25] B. Simon, *Orthogonal polynomials on the unit circle. Part 1: Classical Theory*, AMS Colloquium Series, Amer. Math. Soc., Providence, RI, 2005.

[26] B. Simon, *The Christoffel-Darboux kernel*, Proc. Sympos. Pure Math., **79**, Amer. Math. Soc., Providence, RI, 2008.

[27] D. T. Smith, *Exponential Decay of Resolvents of Banded Infinite Matrices and Asymptotics of Solutions of Linear Difference Equations*, Georgia Institute of Technology, March, 1990.

[28] W. F. Trench, *Linear perturbations of a nonoscillatory second order difference equation*, J. Math. Analysis Appl. **273** (2002), 548–557.

[29] W. F. Trench, *Linear perturbations of a nonoscillatory second order difference equation II*, J. Math. Analysis Appl. **255** (2001), 627–635.

[30] M.-W. L. Wong, *Generalized bounded variation and inserting point masses*, Const. Approx. **30** (2009), 1–15.

[31] M.-W. L. Wong, *Asymptotics of orthogonal polynomials and point perturbation on the unit circle*, J. Approx. Theory **162** (2010), 1294–1321.

[32] M.-W. L. Wong, *Asymptotic analysis of orthogonal polynomials via the transfer matrix approach*, preprint.